Correspondence

A Note on Optimal Multidimensional Companders

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Abstract—**Companding is a widely used method of implementing nonuniform quantizers. There are, however, problems in extending this technique to the multidimensional case. It is known, for example, that not all input probability densities can be quantized optimally in this fashion. In this note we characterize the class of probability density functions that can be optimally quantized with a multidimensional companding-type implementation.**

Companding [1] is a method of implementing nonuniform quantizers. In essence one models a nonuniform quantizer as a zero memory nonlinearity followed by an optimal uniform quantizer in turn followed by the inverse of the first nonlinearity. In [2] a mean-squared error expression is derived for the asymptotic performance (small distortion—high-information rate) to be expected for a multidimensional compander. The function f that maps the domain of the input k-dimensional random vector, having probability density $p(x)$, into the domain of the uniform quantizer is called (for historical reasons) the compressor. The condition derived in [2] for the compressor function to be optimal is that the compressor f must be one-to-one and satisfy two conditions almost everywhere:

1) $f\colon \mathbb{R}^k \to \mathbb{R}^k$ must be conformal, i.e., $\|[f'(x)h]\| = |f'(x)||h|$,

and

2) $|f'(x)| = p(x)^{k/k+2} \Big/ \int_{\mathbb{R}^k} p(x)^{k/k+2}\, dx$,

where $h \in \mathbb{R}^k$ is arbitrary and $|f'(x)|$ denotes the Jacobian of f.

Suppose we consider the problem of finding a function f that satisfies conditions 1) and 2) everywhere. The one-dimensional case ($k = 1$) is easy. Smith in 1957 [3] derives the optimum compressor as

$$f(x) = \frac{\int_{-\infty}^{x} p(\alpha)^{1/3}\, dx}{\int_{-\infty}^{\infty} p(\alpha)^{1/3}\, dx}.$$

Heppes and Szüsz [4] show that when $k = 2$, conditions 1) and 2) can be satisfied if and only if $\log p(x)$ satisfies Laplace's equation. The purpose of this note is to consider the cases when $k > 2$.

Hartman [5], in an extension of a theorem due to Liouville, showed that if $k > 2$ and $f\colon \mathbb{R}^k \to \mathbb{R}^k$ is conformal and continuously differentiable, then f is a restriction of a Möbius transformation. By a Möbius transformation we mean a translation ($f(x) = x + a$), a magnification ($f(x) = rx$, $r > 0$), an orthogonal transformation (f is linear and $|f(x)| = |x|\ x \in \mathbb{R}^n$), a reflection through reciprocal radii ($f(x) = a + (r^2(x - a)/|x - a|^2)$), or a combination of these elementary transformations. A Möbius transformation can always be written in one of the forms [6] $f(x) = rTX + a$ or $f(x) = I(rTX + a)$, where $r > 0$, $a \in \mathbb{R}$, T is an orthogonal mapping, and I is a reflection through reciprocal radii. It is then easy to calculate for the first case $|f'(x)| =$ constant and for the second case $|f'(x)| = \alpha |x - x_0|^{-2}$, where x_0 is a constant vector in $\mathbb{R}^k$ and α is a real constant. We therefore have shown that conditions 1) and 2) can be satisfied everywhere for dimensions greater than two only if $p(x) =$ constant or $p(x) =$ constant $|x - x_0|^{-(k+2)2/k}$. We make the following remarks:

a) In [2] f was constrained to be a mapping from $\mathbb{R}^k \to X_{i=1}^k[0, 1]$, where $X_{i=1}^k[0, 1]$ is the k-dimensional unit cube. We did not make this constraint in the derivation considered above. We note, however, from a generalization of Sard's theorem [7] that

$$m\left(f(\mathbb{R}^k)\right) \le \int_{\mathbb{R}^k} |f'(x)|\, dx = 1.$$

In addition, since conformal functions map balls into balls, we have that the range space of f is bounded and has at most unit measure.

b) As a matter of engineering practicality, it seems that the result derived above destroys the idea of using companders to implement optimum quantizers in higher dimensions. However, there is still the possibility that forcing conditions 1) and 2) to hold only *almost everywhere* could allow more freedom.

c) In [2] an example is given showing how almost optimal performance can be achieved with nonconformal compressor functions. This currently seems to be the area of greatest promise if one desires to retain the ease and versatility of the companding idea.

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Low-Rate Tree Coding of Autoregressive Sources

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Abstract—**The performance of differential pulse-code modulation and random codes is evaluated experimentally for a range of autoregressive sources, including Gaussian and Laplacian sources of orders 1, 2, and 3**

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