AN ANALYSIS OF THE MATHEMATICAL MODELING PROMPTS IN THE ILLUSTRATIVE MATHEMATICS ALGEBRA 2 CURRICULUM

by

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ABSTRACT

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In this paper, mathematical modeling prompts from the Illustrative Mathematics Algebra 2 curriculum are analyzed to determine how well they represent mathematical modeling. Select samples are presented here for an in-depth analysis where each component of the Illustrative Mathematics' own modeling analysis is assessed. The modeling prompts are also assessed on their authenticity based on criteria from Tran and Dougherty (2014). There is a discussion of how well the curriculum prepares the teacher to present the modeling prompts to their class and how well the prompts cover the Common Core State Standards for Mathematics they intend to. We find that the prompts are well constructed and deliver diverse mathematical modeling situations that are authentic and range from a low-level to a very high-level authentic modeling. The prompts accomplish this, the incorporated CCSSM are covered less thoroughly due to the open-ended nature of the modeling process. We also conclude that the high-quality nature of the modeling prompts and the high demand of a modeling situation lend themselves to be presented to students by a highly skilled instructor. Suggested by these findings, the instructor should take great care in assigning the prompts, and schools or districts should provide professional development focused on the instruction of these modeling prompts.

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CCSS	Common Core State Standards
CMF	California Modeling Framework
IM	Illustrative Mathematics
NCTM	National Council of Teachers of Mathematics

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Chapter 1: Motivation

1.1 Introduction

Mathematical modeling can be simply summarized as using mathematics to solve real-world problems. However, it is much more. “Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions” (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, Common Core State Standards for Mathematics). This definition still requires a lot of dissection and analysis to see how it applies to the mathematical concepts in an Algebra 2 course. In this paper we will analyze how well the Illustrative Mathematics Algebra 2 Curriculum tackles the concept of mathematical modeling and provides ideas about potential modifications or required actions a teacher may make for students to reach a high level of mathematical modeling competency.

1.2 What Mathematical Modeling Is, and Is Not

The education world is full of buzzwords, and one could be forgiven for assuming that sometimes new words or names are given to old concepts. A fresh coat of paint can be applied to old ideas with little change. While it is true that mathematical modeling is not a new concept, the emphasis given to it in the Illustrative Mathematics Curriculum and specificity about what it is, is important to understand. The Mathematics Framework for California Public Schools clarifies that modeling is not using manipulatives to represent mathematical concepts, simply calling a graph or equation a model, or using real-world context to solve a math problem (California Department of Education, 2013). Mathematical modeling is a deeper look at solving real-world problems. The context is not there for show. Mathematical modeling uses context as the starting point and then asks students to figure out what they need to do to help make the context make sense and make a decision. One description of what modeling should be is that the problem should begin with an actual real-world problem and one that is solved with mathematics (Usiskin, 2015). In this sense many typical word problems in math textbooks are quite contrived and fall short of this requirement. There is a threshold of authenticity that a truly well-crafted mathematical modeling problem should reach (Tran & Dougherty, 2014). Students should be able to see that the problem is real, and it may not have all the steps laid out in an effort for the problem to fit the mathematical scheme or concept of current study in the classroom.

Along with the authenticity of the real-world task involved in mathematical modeling, another component is that the students should not be focused solely on solving a problem or getting an answer. The problems, solutions, and student responses should be open-ended and may not even be completely explicit in the instructions. The Common Core State Standards for Mathematics highlights this in its description of modeling stating that students should “analyze empirical situations, to understand them better, and to improve decisions” (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, Common Core State Standards for Mathematics). What this means is that an important aspect of modeling is that it should be used to actually do something. Students should be required to make a decision or recommendation based on the data or situation they are presented with.

The Common Core State Standards Initiative goes even deeper with its description, explaining that there is a creative element to modeling (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, Common Core State Standards for Mathematics). These modeling problems should not be cookie cutter examples of how to apply a procedure, and there may be more than one correct path. There may also be more than one correct result based on various valid approaches. Students must make decisions about which mathematical tools could be used. They need to determine the specific type of answer they should provide and how precise they need to be in addition to identifying the variables and data they are given (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, Common Core State Standards for Mathematics). Because these modeling problems can range in complexity, depth, content, and context it is important to consider their purpose and benefit to the students attempting the problems.

1.2 The Process of Mathematical Modeling

As mathematical modeling is a representation of what actually occurs in situations where mathematics is required, it is not a linear, single-step process. It is an iterative process and a natural way of thinking that can lead students to the necessity of mathematics and not the other way around. Many scholars use a graphic representation of this thinking process similar to what is presented here from the GAIMME (Guidelines for Assessment & Instruction in Mathematical Modeling Education) report shown in the figure below (Bliss et al., 2019).

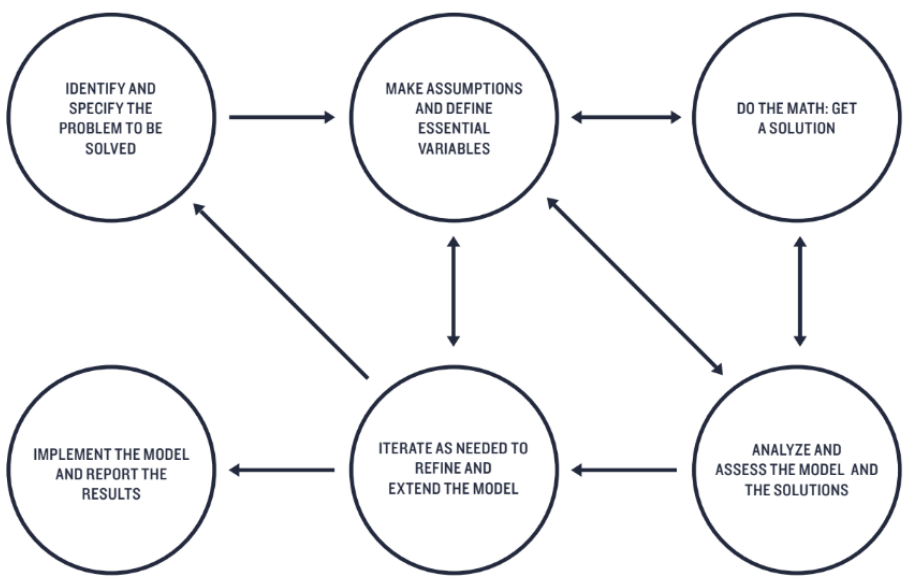

Figure 1: Cyclical modeling process (Bliss et al., 2019)

This description of modeling shows, in detail, how the thought process of solving a real-life problem might occur. Modeling is a cyclical process. Once a question is declared, whether given or discovered, a modeler must seek out and investigate the variables in play. They then need to work out some relationship and attempt a mathematical model. The model is then assessed and the modeler must return to the assumptions they made and make modifications, refine the model, and repeat the cycle until they are satisfied with the outcome or the outcome has reached a point of stability.

The GAIMME (2019) report presents the following as an example of a good modeling problem for the high school level:

> "Michael Jordan is a basketball player from North Carolina. I was watching him play on television one day. As he drove for the basket, he was fouled. The announcer stated that "Michael Jordan is making 78% of his free throws." He missed the first shot and made the second. Later in the game, Michael Jordan was again fouled. This time, the announcer stated that "Michael Jordan is making 76% of his free throws." Determine the number of free throws Jordan had attempted and how many he had made at this point in the season" (p. 65).

The heart of this problem is that students need to choose a method for calculating the free throw percentages and then analyze the validity of their solution. It is a good example of a modeling situation because the problem is open-ended and requires students to "explore, to fail and regroup" (Bliss et al., 2019, p.64) The problem also still requires a delicate balance of teacher input to help students question their own methods and the meaning of their conclusions. It has the authenticity required of a good modeling problem as well. Students, likely, will have seen a sporting event in which numerous statistics are casually mentioned. This problem relies on student experience with the context but goes deeper. Students must work to find the information they need from the problem or elsewhere to fully elicit a suitable model. This good example of a modeling problem also highlights a skilled instructor to help guide students through the modeling process and to a satisfactory response. Students may be content with their initial thoughts or model, but the teacher could help them back to their assumptions to see flaws or omissions. The foundation of a modeling problem is the problem itself, but without requisite teacher support, even a well written problem may fall short.

Bad modeling problems are overly simple or lack authenticity. An example would be a simple word problem that inspires students to apply a known mathematical concept. One classic problem type from geometry would be:

> At a certain time of day, a tree casts a shadow that is 20 ft long, at the same time a 6 ft tall man casts a shadow that is 3 feet long. What is the height of the tree?

While this example may have value in getting students started on deciphering context and putting it into a diagram and determining the needed mathematical procedures to solve, it lacks authenticity. It may involve real things, but it is quite contrived. For an authentic modeling problem, Tran and Dougherty describe that the event must have a reasonable chance of happening (2014). In this case, while this method may work, it is unlikely to be the method of choice, as it is not practical in many situations. It also does not involve any decision-making for students. They do not have to decipher complex data, nor do any research of their own. The question is not open-ended and has *one* calculable solution. It fits neatly into a similar triangles box and requires a simple application of a known formula. The teacher's role in this problem would be to cite previous knowledge about similar triangles and diagrams, but there is no modeling process needed. This example also lacks the essence of a true modeling problem.

1.3 Importance of Mathematical Modeling

There are a range of skills and concepts students must master. Modeling is important as a tool for helping students “understand the underlying principle of mathematics” (Sokolowski, 2015). Modeling can help students take concepts they are familiar with and see how they might arise naturally in a problem-solving manner. This is different from applying concepts to rigid, contrived application problems. Tran and Doughtery (2014) describe how the authenticity of the problem in addition to the modeling process allows students to demonstrate Standards for Mathematical Practice (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, Common Core State Standards for Mathematics). In this immersive way, students are engaged in mathematical thinking throughout the task.

The in-depth nature of mathematical modeling also promotes many of the Common Core State Standards Initiative's Standards for Mathematical Practice. These standards have implications for the long term in mathematics education (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, Common Core State Standards for Mathematics). When students are exposed to legitimate modeling problems, they not only get a glimpse into how mathematics is used by real people, but also exposure to the thinking process of mathematics that can make for better overall reasoning skills and mathematical skills. One example is a small study of high school students exposed to modeling problems in the area of optimization. In this study the researcher concluded that the modeling process helped students more thoroughly understand the principle they were studying and improved their thought processes on the content in how they arrived at their solutions

(Sokolowski, 2015). This deeper understanding highlights the importance of modeling for students that may be aiming for a STEM field of study. It also demonstrates how modeling may have implications across curriculum by extending student thinking process or angle of attack in any problem-solving situation. Indeed, mathematical thinking extends beyond mathematics, but mathematics is also sometimes hidden in other subjects that may be less obvious. One area where modeling is very useful across curricula is in statistics. Modeling with statistics has a place in many areas of study, as well as in simply understanding the world around us. The Pre-K–12 Guidelines for Assessment and Instruction in Statistics Education II (GAISE II) report underscores the importance of statistical fluency stating that data is all around us and a fluent person may be able to advance their career paths (Bargagliotti, 2020). Modeling can produce not only better mathematical understanding, but also problem-solving mindset that extends beyond mathematics, or at least helps us see mathematics in broader situations. In a study of two elementary classrooms, Suh et al. found that even at this level students engaged in problem solving and learned from their mistakes (2017). This also led to the students engaging in skills associated with creativity, communication, and collaboration while working on modeling. These skills indeed extend beyond the strictly mathematics classroom context, and at the high school level, it may be even more important that students nearing adulthood should have experiences practicing these skills.

Finally, and critically, a good modeling prompt allows students to gain experience in almost all of the CCSS Standards for Mathematical Practice. The standards for mathematical practice included in the CCSS include making sense of problems, persevering in solving them, constructing arguments, using appropriate tools, reasoning abstractly and quantitatively, attending to precision, and modeling itself (National Governors Association Center for Best

Practices & Council of Chief State School Officers, 2010, Common Core State Standards for Mathematics). These practices are based on the NCTM Process Standards and are written to help guide teachers instruction and student learning so they can make connections across seemingly discrete mathematical content strands and build deeper understanding (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, Common Core State Standards for Mathematics). Modeling, as defined, requires students to analyze a problem, define variables, and work to develop an appropriate model to make predictions. The modeling cycle also asks students to refine their model and reassess their choices. These are exactly what is described in the Standards for Mathematical Practice. Through modeling, students make sense of the problem, use tools (graphing software, simulation software, or physical models and measurement tools), and finally persevere in solving the modeling task.

1.4 Illustrative Mathematics

Illustrative Mathematics was originated in 2011 with a goal of demonstrating what a student would have to do to meet the CCSS in mathematics through direct examples (Tumarkin, 2019). It has since become a full-fledged curriculum. According to edReports, the IM curriculum meets expectations in its alignment with the CCSS in mathematics in the areas of focus and coherence, rigor and mathematical practices, and usability (edReports, 2019). This implies that the curriculum should do well with meeting the math practice standards of the CCSS as well as many of the mathematical content standards included.

As part of the curriculum the authors included specific mathematical modeling prompts beyond what is included in the standard sequence. These prompts are not contained within the main sequence of coursework. To assign these prompts the teacher must seek them out and craft a unique lesson tailored to the modeling experience. There are nine modeling prompts that have a varying structure but include aspects such as the overarching question, the data students will need, and guiding questions to lead them to solutions. What sets these modeling prompts apart from traditional coursework is that they are intended to fit the modeling process. They do not include all the aspects in every case. Each prompt has a unique setup and a unique mixture of information that may or may not be present in a modeling situation. Students attempting these prompts are exposed to different situations at varying degrees of complexity.

Chapter 2: Methods

2.1 Procedure (What I'm looking at and how)

The structure of the Illustrative Mathematics Algebra 2 is such that each unit has one or more specifically designated modeling prompt. Each modeling prompt has multiple versions, designed at higher and lower levels of modeling. The modeling prompts are not embedded into the main curriculum and are enrichment materials. The sequence of the curriculum would not be thrown off by the omission of these modeling prompts. They are often disconnected from the concepts in the preceding units. Additionally, in the teacher guidance for the curriculum the authors note that there is modeling embedded throughout the standard sequence, however it is "scaled-back" and students need only engage in "a part of the modeling cycle" (Kendall Hunt, 2019). The goal of this analysis is to determine how well the specifically designated modeling prompts hold up against the stringent descriptions and definitions of mathematical modeling. The analysis will consist of the following aspects of each modeling prompt: The teacher's role, the prompt's rating on the California Mathematics Framework Spectrum of Mathematical Modeling (CMF), Illustrative Mathematics'(IM) own rating called a lift analysis, Common Core State Standards (CCSS), and Authenticity.

I will evaluate the teacher's role in the mathematical modeling prompt by analyzing provided teacher materials, determining how much information or background the teachers may have to give students, and the number of leading directions they are asked to give students. Often the context of a modeling prompt requires activating some prior knowledge or discussing the problem so that students fully understand the situation they are being asked to investigate and model. There is a risk that the situation described is too foreign to students' experience and, while this may be a good thing, it could also detract from the experience. Focusing on how much the teacher may need to give students, and how heavy the teacher's role may be in guiding students, will give insight into the overall modeling quality of a prompt. Ideally, a modeling situation arises naturally and inspires curiosity. While this is not necessarily the case in a classroom setting as students are still learning the process, the role the teacher plays can influence how much of the modeling process students take from the assignment. Given the curriculum's suggested teacher's role, one can assess the nature of the guidance a teacher is asked to provide and determine if the prompt requires too much involvement or not.

A more objective measure of the prompt's quality is The California Mathematics Framework Spectrum of Mathematical Modeling. The framework provides a list of nine levels of mathematical modeling. Level one, the lowest level, is described as a math problem but not actually a modeling problem. At levels two and three, there is real-world context, but the students are still expected to use known mathematics to solve a specific solution. By levels four and five, the problems are more complex and may offer multiple pathways or expect the student to check their solution for its validity. Level four allows for the sequence of questions to guide the student to the desired solution. Level five is similar but less guided and students can begin to make choices about their solution. Level six describes the beginnings of true, independent modeling. The students are required to make choices, assumptions, and assess their models. Problems at this level still involve teacher guidance in making those assumptions or guidance from the prompt itself. At levels 7 and above we begin to see more descriptions of authentic modeling eventually reaching level nine, the highest level. Level nine would be equivalent to a true modeling situation in which students are in complete control of the context, question, and methods (California Department of Education, 2013). It provides a great description of the overall problem's modeling level, but it may miss some small points or variations intended by the authors to make a problem more accessible or complex in a given context.

To help with these differences and intricacies where California's modeling framework holistic score falls short, especially between the various versions on each prompt, Illustrative Mathematics includes its own analysis of the modeling prompts. The IM curriculum provided lift Analysis rates five distinct aspects of modeling and will help to differentiate the version of the same prompt. The Modeling tasks are given a score from 0-2 in the following categories: Defining the Question, Quantities of Interest, Source of Data, Amount of Data given, and The Model (Kendall Hunt, 2019). The analysis then provides a simple average to apply a comparative overall lift. In general, a low score indicates a more highly structured problem in which students are given more information and less freedom to choose paths. A higher rating implies that the prompt is a more genuine modeling problem with more opportunity for student choice in developing their response to the prompt (Kendall Hunt, 2019). It also allows the teacher to select for groups to work on appropriate modeling prompts based on their levels of proficiency with the process. This lift analysis will also be evaluated under the same scrutiny of definition of good mathematical modeling to test the veracity of the author's claims.

The authors of these prompts clearly put a lot of thought into modeling and have already done their own analysis by producing this lift score. Digging deeper into their estimations of their own work to provide validation or repudiation is a logical and necessary aspect of the analysis here. Understanding each of the five aspects in the lift analysis is crucial in understanding the modeling prompt.

What the authors mean by Defining the Question is if the question is posed clearly and whether there are elements of ambiguity. Students need to have a clearly defined task and while an amount of ambiguity can be the mark of a good modeling situation, the authors rate this aspect of the questions based on how the information is leading. If students are led to make certain assumptions, then the prompt may be rated lower. At the highest level students have more freedom and can make their own choices with little guidance from the prompt itself.

The quantities of interest are essentially the moving parts of the problem. The variables involved can either be clearly stated or left to the student. An example would be asking a student to model a relationship between the height and the velocity of a falling object. In this setting the variables are clearly stated, and it would receive a low lift score. In contrast, a setting in which the data may be given, and students can choose to select velocity and height as their variables would score higher.

Before defining the variables, it would make sense to determine where the data students are using is coming from. The Source of Data is also scored, and it is essentially a rating on whether the data is explicitly provided or if students must seek it out on their own through research or experimentation. In some cases, there may be a combination of these efforts to complete the task.

In addition to Source of Data the Amount of Data is considered. While closely related, this rating applies when the modelers are provided with too much data that needs to be sifted through, or not enough that they would need to conduct the experiment or research. The modeler must make decisions about how to use or find the data required. If the students are given the exact and sufficient amount of data required to complete the task, the prompt would be rated low. If the task of sifting through data or seeking out data is more challenging the prompt receives a high lift.

Lastly, the modeling task is scored on how directed the mathematical model required is suggested. If students are told to make a specific type of model the lift score would be lower. If the students must decipher the type of model or have choice in the type of model, the lift score would be higher. At the highest-level students might have to apply their mathematical thinking to graphs or diagrams before they can determine what type of model they need to make.

Another aspect of modeling in need of analysis is confirming that it actually contains mathematics. Much of mathematics is the thought processes that are required to evaluate and think critically about a problem. The CCSS describes these thought processes in the standards for mathematical practice. Mainly through:

- “MP1: Making sense of problems and persevering in solving them. The standard states that students should explain the problem to themselves and make conjectures about aspects of the problem” (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, Common Core State Standards for Mathematics).
- “MP2: Reason abstractly and quantitatively. What this means is that students compare and take the context given and represent it symbolically, which is essential in a modeling situation” (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, Common Core State Standards for Mathematics)”
- “MP3: Construct viable arguments and critique the reasoning of others. As part of the modeling process students need to justify their results and process as described in this standard” (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, Common Core State Standards for Mathematics).

- “MP4: Model with mathematics” (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, Common Core State Standards for Mathematics).

While the standards from mathematical practice should be present in any modeling situation if it is indeed a modeling prompt, any other mathematics standard the authors suggest will be assessed. A good modeling problem should also contain mathematics in the form of skills and concepts. Using the CCSS this aspect of mathematics can be assessed; however, the process of modeling can be varied enough so that some skills the authors think a problem requires may be omitted by the modelers. Indeed, the concepts identified by the authors will be required to make the standard model implied, but there may be alternative methods used by students to yield similar results. An example would be a prompt designed to encourage students to build a specific type of model, however, a similar model may exist using different properties. An exponential model may suffice in place of a polynomial model. These will be evaluated with the CCSS and will be unique to each problem, but several standards should be present for a high-quality modeling problem.

Finally, I will assess the level of authenticity of the prompt. As this can be a subjective feature, I will consider authenticity as something the students will appreciate or not. This means that the problem should be relevant and meaningful to the students. If the problem is outdated, for example, it will be less meaningful. A problem revolving around long distance phone call rates would not be an authentic modeling problem in 2025. Torulf Palm described criteria for evaluating the authenticity of a word problem which was summarized by Tran and Dougherty (2014) in a table of six criteria.

The Six Criteria for Authentic Tasks	
Characteristic	**Description**
Event	In a simulation of a real-life task, the event has taken place or has a fair potential of taking place.
Question	The question is one that actually might be posed in the event.
Purpose in the figurative context	The purpose of the task needs to be as clear to the students as it would be if they actually encountered it in context—either provided explicitly from the description or implicitly from the context.
Information or data	The task describes specific subjects, objects, and places, includes accessible data, and provides identical or close numbers and values as in the simulated situation.
Language use	The task does not, for example, include difficult terms that hinder the students in the solution process if the corresponding difficulties do not occur in the simulated situation.
Tool demands	The tools and guidance in using them have to be reasonably the same in the simulated problem and school situation; mathematical knowledge and skills that students need for solving the task are available.

Figure 2: Six Criteria for Authentic Tasks from Tran and Dougherty (2014) adapted from Palm (2008).

To the extent that it can be evaluated objectively, I will use the six criteria to help prove a sense of the authenticity of the prompt. Tran and Dougherty (2014) describe the authenticity of the event as how the situation relates to a real-life task. This means that if the task is to be authentic, it should be reasonably assumed to be possible and/or actually taking place in the real-world. Next, the authenticity of the question is considered. The question needs to be connected to the event and not simply contrived to fit the mathematics. A classic example is the problem of applying real mathematics to a context in which no person would do such mathematics, such as determining the angle a ladder makes to reach a window. This type of problem can be a less dull way of using trigonometric ratios but has no practical purpose. Additionally, the purpose of the question is important and another one of the six criteria. The question should clearly follow from the context. When considering the data another criterion is that the task involves real data or simulated data sufficiently close to the real data so that the resulting models are valid and cannot lead to confusion such as resulting in negative values where none should exist. The language used should also be appropriate and not overly technical.

While the modeling should be real-world, the students are learning, and word choice should be reasonably appreciated for the grade level. The final criteria is that students should be using the tools that someone developing the model would use. This does not have to be a piece of expensive engineering equipment but should involve the same level of mathematical expertise to solve the problem. If the real-world modeler needs to measure an angle, the students should be reading the angle. While this measurement will likely involve different tools, the purpose and required knowledge is the same. Authenticity can admittedly be a subjective attribute, but with considerations for the modeling task's audience of Algebra 2 students, one can assess the task using these criteria and compare relative authenticities between prompts.

Chapter 3: Analysis

3.1 Unit 1 Evaluating a Sample Response to a Modeling Prompt

The first modeling prompt is a training task. The first modeling activity is designed to teach the modeling process to the students. Students do not have to create a model, but they have to analyze the modeling process in a sample student response. The goal is to make sure students see what the modeling process may look like and pre-load their thinking with an example of the iterative process. Essentially the goal is to *model* the modeling process. In this way the first prompt is a training ground for the expectations in the following units. This section is complete with a solution for the students to discuss. It gives a window into the type of rigor and thoroughness that the authors expect for a modeling prompt response. What follows will be a brief description of this task as it may help inform the reader as to how the modeling assignments will be conducted in the classroom setting.

3.1.1 Teacher's Role

In the teacher materials the book describes that the teacher should explain what a modeling prompt is and prepare students for what they are going to be asked to do. It gives definitions of what modeling is and uses descriptions such as, "Modeling prompts are also often expressed in words, and involve more thought than a more straightforward exercise" (Kendall Hunt, 2019). This aligns with our definitions of modeling from the NCTM and CCSS. It also explains the difference between a modeling prompt and a word problem. The teacher materials also delineate the expectations for student work and ask the teacher to lead discussion and take clarifying questions, so students have clear guidance on what they need to do (Kendall Hunt, 2019). The teacher is instructed to provide students with the modeling rubrics and to allow students to discuss or ask questions. These are important steps for students to fully understand the task. Modeling problems may be designed to be open-ended, but students should be aware of their structure. The final piece of the material provided is a list of advice on modeling for students.

This advice is a nine-point chart similar to modeling task structures outlined in the GAIMME Report but written in a detailed accessible way for the students to reference throughout the modeling process. The chart reminds students of the cyclical nature of the modeling process and provides bullet points containing questions for students to ask themselves about their work. Some examples of how this chart provides guidance are by asking students to consider what information might be useful, to write down any information they would still like to know, and to check their response (Kendall Hunt, 2019). The chart would do well as a poster in the classroom as it outlines the mathematics of the modeling process and gives students reminders on how to keep pushing forward in their task.

3.1.2 Activity

The sample prompt students are given is: "Two friends, Han and Jada, live 7 miles apart. One Saturday, they decide to meet up somewhere between their houses. They each leave their house at 8 a.m. and travel toward each other. They want to choose a place to meet so that they'll both arrive at the same time. Where could they meet?" (Kendall Hunt, 2019).

The sample response students are asked to evaluate demonstrates skills in proportional reasoning and linear modeling through distance-rate problems. It does not directly align to the curriculum contained in the first unit of the text. Though it does not directly follow from the unit it is content students should have seen prior to the algebra 2 level. In order for students to thoroughly evaluate the problem they would need to understand the distance formula. They start by determining speeds at which they think the friends will be walking, then they define variables and write ratios representing the time they will travel given a distance. The sample continues and the ratios are set equal. What follows is a lengthy algebraic sequence. Then the problem is solved by showing each step but not describing each step. Students will have to determine exactly what happened from line to line to verify the solution. The problem walks students through the rest of the reasoning and determines the distance each of the friends will have to travel. The sample problem also highlights that there could have been other choices made, and the three friends could have been traveling at vastly different speeds, which would alter their meet-up distances.

3.1.3 Authenticity

This activity is designed as a learning task. It models what students should be expected to do moving forward with the open-ended tasks ahead of them. This task does not register a high rating for an authentic modeling task. Students may be able to follow the logic in the problem, but if the first criteria laid out by Palm (2008) and summarized by Tran and Dougherty (2012) is that the problem is reasonably likely to happen, then it fails. It is much more likely that friends planning to meet will simply choose a time and a location. The question in this prompt is indeed a question that might be posed by a group of friends, and the sample response includes much of the reasoning a student would consider through the process. As a sample, however, it fails to represent the reasoning and choices that a student would have to work through in order to arrive at those assumptions. The research steps and trial and error are taken away.

3.1.3 Summary

In the end, the problem is to be evaluated by class discussions with the teacher leading students through the modeling process and students evaluating the rubric for this problem. Ideally, this is how each of the modeling prompts for each unit will be done, however, the students will be the ones creating the model and solving the problem.

This problem is a good introduction to the process but may be a bit mathematics heavy for students unfamiliar with the distance-rate problems. Students should have been exposed to this concept, but since it is not contained in the preceding chapter, it may be difficult for them to fully follow the line of thinking. The sample, however, illuminates the process well and students will have a guide and experience seeing how mathematical modeling may look. It also serves as a demonstration to the students that they may need mathematics other than the concepts covered in the preceding chapters to investigate future modeling prompts.

3.2 Unit 2 Modeling Prompt: Viral Marketing

The first modeling prompt students are faced with is about a viral marketing campaign. Their task is to make a choice as to how an organization could attempt to raise awareness through different marketing strategies. Students are provided with choices, and the goal is to determine how each strategy may or may not spread and reach the most people. This prompt comes with three versions at different levels, allowing the teacher to differentiate student groups as needed for their class or assign one version to the whole class based on the teacher's knowledge of their students. To help see the differences in the level of modeling required by students each version has been given a lift analysis score reflective of its modeling difficulty. Along with mathematical modeling, each version is supposed to allow students to demonstrate a level of proficiency on the same set of Common Core Standards for Mathematics, despite the changes in the lift score. If students can develop an effective model, and respond to the question in the prompt, they should be applying the same standards regardless of the version they attempt.

3.2.1 The Teacher's Role

The teacher's role as indicated by the instructional materials for this task is simply to get students thinking about viral marketing campaigns. The goal is to explain what is meant by viral marketing and to make sure students have some concrete examples of viral campaigns they may have seen. Activating this prior knowledge and facilitating a discussion as to why a company or organization may want a viral campaign gives students an entry point into the problem. To maintain the integrity of the modeling problem, the suggested actions of the teacher should not lead students down one particular path or suggest methods for solving. To that end, these instructional materials, while very helpful for the teacher, do not provide any unnecessary ideas to the students and should help for a more authentic modeling experience.

3.2.2 Viral Marketing Highest Lift

This is designed as the highest-level problem. Students are asked to develop a viral marketing campaign for an animal rescue and are provided with three options:

- Send the fundraising message to their most loyal supporters. Ask them to each pass the message to a certain number of their animal-loving friends, along with the same request of relaying it to a certain number of their friends, and so on.
- Send the message to all supporters on their list. Ask them to share the message with everyone in their network of friends, along with the request to continue passing the message to everyone on those friends' lists. With this strategy, the organization recognizes that only a fraction of the recipients are likely to read the message or pass it along.
- Broadcast the message by buying a commercial that will air on the local news.

Figure 3: Viral Marketing (1) (Kendall Hunt, 2019)

Students need to decide between the options and are given very little direction. Students must make assumptions, research data on the methods, and put together a proposal and justification for their mathematically justified best option. The authors rate this task as the highest level and provide the following lift analysis.

attribute	DQ	QI	SD	AD	M	avg
lift	2	2	2	2	2	2.0

Figure 4: Lift Analysis Viral Marketing (a) (Kendall Hunt, 2019)

It has received the highest possible score in each category. This means the authors believe this prompt requires students to reason through and determine the question they are truly responding to. They have to determine the variables while researching what else may be necessary. There are few directions given and students have the freedom to take this task in any direction they desire. While the modeling levels for each aspect are high, the teacher may still need to guide students in making choices to maintain the highest level of thought and modeling. If students do not dig deep enough, they may only scratch the surface of the problem. There are variables in this problem students may not think of.

DQ-Defining the Question

In this version of the prompt students are given the question, "Which strategy would likely yield the better outcome?" While this question is very clear, it does not define "better outcome." Students are free to make assumptions about what the organization may want from the marketing campaign. Students are free to determine what would be better. Since students are given the question, this prompt falls to level 7 on the California Mathematics Framework Spectrum of Mathematical Modeling. The framework describes modeling questions in which the teacher, or in this case the textbook, has given the context and stated the question (California Department of Education, 2013). It is still a high level of modeling, but because students have been given a situation and a question it is not the highest level of modeling according to this framework.

QI-Quantities of Interest

The information in this problem is very minimal. Students are given the goal of a viral marketing campaign and three strategies, but there are not numbers associated with the strategies. Nor are there any hints as to how those strategies should be applied or represented mathematically. It is because of this that the lift score is so high. Students must think mathematically and may even have to trial several different models before arriving at a usable structure to see the patterns the authors hoped they would see. Again, this problem lands at a level 7 on California's Spectrum. Students can ignore or simplify and determine relevant variables in order to find their representative model (California Department of Education, 2013).

SD-Source of Data

There is no data cited in the problem so students will have to determine what, if any, data they would need. Once they have decided what a better outcome means, they will need to decide what data and sources are available to help them make a decision. This idea maintains the concept of high-level modeling and mirrors how a person may institute the modeling process outside of the classroom. Data is not always readily available. To thoroughly respond to this prompt, students will need careful research.

AD-Amount of Data given

The high score in this category is due to the lack of data cited by the authors of the problem. Students will have to make assumptions and trial likely values throughout the process of making their model. They may even need to look up information on the spread of viral videos. Again, this would fall firmly in level 7 of California's Spectrum. The AD category is closely related to the SD category. In this prompt the complete lack of data means that during the students' research they will need to determine the point at which they have sufficient data for their model. This is another important aspect of modeling students need to learn about. How much data is enough, and when is it overkill?

M-The Model

In this version of the problem, students are not told how to construct a model or how to apply any of the data they have found. It is because of this that the prompt gets the highest score in this category. Students are free to develop their own model and apply any of their mathematical tools to the process. This also lands at level 7 on California's Spectrum. In this high-level modeling situation, the models that may be developed could be very different and that would make comparing the results from different models an even more interesting activity.

3.2.3 Viral Marketing Medium Lift

The curriculum has provided a second version of the same modeling situation. In this version more information is given, and concrete numbers are associated with the choices in marketing campaigns.

- Send the message to 20 people, and assume that each person will share the message with 3 other, new people (and each of them will share with 3 new people, and so on).
- Send the message to 5 people, and assume that each person will share the message with 6 other, new people (and each of them will share with 6 new people, and so on).
- Broadcast the message by paying for a television commercial that reaches 100,000 people, but can't be shared easily.

Figure 5: Viral Marketing Choices (2) (Kendall Hunt, 2019)

Students are given the number of people a message may be sent to and the estimated number of people that the message may then be forwarded to in each of the three choices of viral marketing method. The details and information about each of the three choices of marketing campaign are the limit of the differences in this prompt. Students are still required to determine which method is best with no other differences from the first version. The lift analysis is provided below.

attribute	DQ	QI	SD	AD	M	avg
lift	1	1	0	0	2	0.8

Figure 6 : Lift Analysis Viral Marketing (b) (Kendall Hunt, 2019)

This prompt ranks much lower in every category, save for the model developed. Since students are given leading information, sample data, and a structure to start building a pattern for how the viral video will spread, they need to make less assumptions. They do not need to seek out any new data. This puts the problem at level 6 on California's Spectrum because students are guided by the prompt with cited data. Students are still required to make a model and hopefully follow the modeling process to determine if their model makes sense, then revisit it to adjust as necessary.

DQ-Defining the Question

The question here is very similar to the first version, however students are told to base their decision on the given assumptions. Students have less freedom to make a choice in what determines the better outcome. The way the question is posed still allows students to truly think about what the best method will be, and they still have a choice in how they apply the provided data. In this way the prompt has not been completely deprived of its modeling potential.

QI-Quantities of Interest

The quantities are given in this version. Students have a set of values to work with and develop their model. By the design of the question, they are stuck using these values as their assumptions. They do not need to think of and determine other variables in order to get to a model and make a decision on which method is best. There is a level of the question that students can make their own assumptions, and they can modify or add variables to add complexity. The information is not completely thorough as to minimize the modeling process.

SD-Source of Data

The source of the data in this version is the problem itself and the required data is explicitly given. The low lift score here is due to this fact. There is no need to seek out new data or make assumptions. Students can begin constructing models for each of the three options almost immediately.

AD-Amount of Data given

The amount of data given, while not a lot in quantity, is completely sufficient for the students to begin and complete a response to the question. There is no need to seek out or sift through any amount of data. Because of this, the score is zero in this category.

M-The Model

If the goal of the modeling prompt is to simply have a functioning model, students can still get there with the information given in this problem. They still have an opportunity to work though the information and develop their model, and touch on the CCSS associated with the prompt.

While this version is a much lighter lift, it does give students a valuable opportunity to take information and turn out a functioning model, choose the best method, and justify it with the given information. Not all students may be ready for a full modeling experience and this differentiated approach could help steer some students in the right direction.

3.2.4 Viral Marketing Low Lift

attribute	DQ	QI	SD	AD	M	avg
lift	0	1	0	0	2	0.6

Figure 7: Lift Analysis Viral Marketing (c) (Kendall Hunt, 2019)

In the final version of the prompt on viral marketing, the problem has been simplified even further. The two areas of note are that the question is more specific, and the data provided is more explicit. The authors have included an extra aspect of time, directly into the choices.

- Send the message to 20 people, and assume that each person will take 1 day to share the message with 3 other, new people (and each of them will share with 3 new people, and so on).
- Send the message to 5 people, and assume that each person will take 1 day to share the message with 6 other, new people (and each of them will share with 6 new people, and so on).
- Broadcast the message by paying for a television commercial that reaches 100,000 people, but can't be shared easily.

Figure 8: Viral Marketing Choices (3) (Kendall Hunt, 2019)

Unlike the first two versions, the question has changed to a specific question about how many people have seen the message by the sixth day. Putting these parameters on the students' solution removes an important part of modeling from the problem. Students should be using their model to make a decision. In this problem they are simply answering a question and do not have to stand behind their decision. While they could surely be asked to justify their answer, they have only stated an objective fact according to their chosen model. The data provided is also so specific that a discerning student may recognize it as the building blocks to write an exponential function or a geometric sequence. Without specifically stating what their model should be, the authors have given a lot of information, and this version of the prompt aligns with the CMF at level five. The teacher has provided a simplified version and students may develop differing models. The problem has been specifically structured to elicit a solution based on very specific data (California Department of Education, 2013).

3.2.5 CCSS

In each of the prompts students can show proficiency on several CCSS. Once the data is given, or assumptions have been made, students are faced with a decision on how to apply that data and create a mathematical model. This problem lends itself to geometric sequences, arithmetic sequences, and models represented by functions. A strength of modeling tasks is that students are required to apply their knowledge independently or even seek out knowledge that they are missing. Students may come to a conclusion that they need a mathematical tool they are not familiar with, and it is a learning opportunity. In order to make the claim that CCSS are present in the problem at hand, we assume the skill represented in the standard is what would be required to thoroughly model the prompt. We also must take into consideration that students may find alternative methods and the standard may not be applicable. What follows is a list of the standards the authors expect are embedded in all three versions of this modeling task and a short description of how prevalent or apparent that standard is in the task.

- HSF-BF.A.1 “Write a function that describes a relationship between two quantities” (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, Common Core State Standards for Mathematics).

Whether given the data, or looking up, or even making some educated assumptions, if students have arrived at a model for the situation, they will have met this standard. Where each of these prompts falls short is that it may be possible to arrive at a solution without developing a function in the formal sense. Students that have made a table of values and stopped at a reasonable point, may still be able to determine which method will result in the best outcome. While a table can be a representation of a function, it is not sufficient to demonstrate that one understands the connection to the function concept.

- HSF-LE.A.1 "Distinguish between situations that can be modeled with linear functions and with exponential functions" (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, Common Core State Standards for Mathematics).

Students responding to any of the three prompts have an opportunity to show they recognize that the different assumptions will yield different patterns. There should be a clear depiction of an exponential relationship in any of the patterns students find from the first two assumptions. While there is not an opportunity to write a linear model with the information given, any students attempting this will provide a teacher with important information about their understanding of the differences between linear and exponential relationships.

- HSF-LE.A.2 'Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs" (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, Common Core State Standards for Mathematics).

If any student or group arrives at a final model, it should be exponential. All three versions give students the chance to show that they can create models. Whether they have looked up the data themselves, the spread of a viral video should follow this pattern. The key issue is recognizing how the viral marketing campaign works. If it starts with five people each sharing with six others, and that pattern continues, students should recognize the repeated multiplication. They should conclude that it is exponential and seek out a model that reflects this nature.

- HSN-Q.A.2 “Define appropriate quantities for the purpose of descriptive modeling “(National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, Common Core State Standards for Mathematics).

Regardless of the prompt students will need to make sure that they have defined the variables and attached the appropriate meaning to them. While students attempting the highest lift version will have to discover these quantities through research or assumptions, and students with the other prompts are given some of that information, all parties will have to make sure they have defined them appropriately to defend their models. In each prompt the deserted goal is to spread the message as far and wide as possible. Students researching or choosing an option are led to the idea of getting as many people informed over a period of time, thus determining the two variables in question are number of people and time.

3.2.8 Differences Between Versions

There are clearly differences between the task versions in terms of mathematical modeling. The tasks, however, allow students a similar chance to demonstrate the mathematics in the CCSS. In each version students can show levels of proficiency with the mathematical skills stated in the standards. There are no discernible differences other than the higher-level modeling version will require the students to synthesize the mathematics contained in the standard as opposed to being directed to perform a task designed to elucidate that standard. The idea of synthesizing mathematics is more closely related to the concept of modeling and should not detract from the understanding demonstrated for each specific standard.

3.2.7Authenticity

The Viral Marketing prompt has a high level of authenticity according to Tran and Doughtery's (2014) summary of the criteria. The event of marketing occurs regularly, and high school aged students would have been exposed to multiple marketing campaigns outside of the classroom. The modeling task also describes the actual methods used by marketers. The language used is completely in line with high school expectations. Lastly, the question posed is exactly what a company would be considering in this type of situation. The ideal goal is to reach as many people as possible.

3.2.8 Summary

This is the first modeling prompt in the Algebra 2 sequence that students would have to do themselves. The highest lift prompt is the highest lift possible according to the author's lift metric. Teachers are able, however, to assign the problem at different levels so that students are able to access the non-modeling mathematics underlying the prompt. Even in the lower lift students gain experience with modeling. The task never falls below a rating of 5 on the CMF. This means that even at its lowest level it is approaching true modeling. There are still choices the students must make, and students have the chance to develop a model. At the highest level presented in the problem students are able to model at a level aligned with the California Framework's level 7. This means that in order to make the model students are required to make assumptions, assume variables, and make choices on which data to include or ignore. Even though the highest level of the Viral Marketing prompt has the highest lift rating, it fails to reach the highest level on the CMF. The CMF's highest level is a true mathematical modeling situation in which students have full control of their question and decisions. It fails in this regard because the students were provided with the context and the question to consider. This choice in context and question are required for the highest levels of modeling.

The viral marketing prompts all represent authentic situations and have the teacher's role clearly defined. The event is one that could, and does, happen in real-life. The teacher's role is to guide students through this modeling process and the curriculum materials do not offer excessive suggestions.

3.3 Modeling Prompt 7: Swept Away

While the first modeling prompt is indicated to be high-level modeling, and differentiated to allow students multiple access points, most of the remaining prompts fall into a lift average just above one. Only one other offers the highest overall average lift of two. Prompt seven, at its highest lift, serves as an example of a middle level lift prompt. In the task students must choose a path and make assumptions to get the final model. The context, data, and the question are provided for the students. The guidance given varies between the versions, but it still maintains this higher level in the CMF. The task describes the tides in Boston over a certain time period, it includes a table of values and then asks that a model be developed to describe the water level on the given dates. The task also includes follow-up questions that are applications of model and checks for accuracy. In the two versions of this prompt, students are given all data necessary to make a mathematical model of tide depth. Since the data is provided, the students need not make many assumptions or research other variables. There are differences in the amount of guidance students are given, and the overall modeling levels of both versions are different, allowing for a teacher to differentiate groupings.

3.3.1 The Teacher's Role

The text asks the teacher to begin with a discussion of tides and what causes them. Students are given a chance to posit their own ideas, but the authors suggest that the teacher circles back and discusses the role the moon plays after students have made their model. The data given in the prompt is specific to Boston, MA, however, because of the structure of the questions, a teacher could allow students to choose their own data. The authors do caution that many tides are more complicated than the ones provided (Kendall Hunt, 2019). This is a situation where the teacher is making the decision and providing a simplified version to guarantee access to the modeling based on student skills. The provided data fits a trigonometric model quite well and the location of this prompt within the larger context of the curriculum is at the end of the unit on trigonometric functions.

3.3.2 Swept Away Higher Lift

This modeling prompt is given the following lift analysis by the authors.

attribute	DQ	QI	SD	AD	M	avg
lift	2	0	0	1	2	1.0

Figure 9: Lift Analysis Swept Away (a) (Kendall Hunt, 2019)

Both versions of the prompt also provide a table of values for the hours after midnight and the water level in feet for a period of 48 hours.

DQ-Defining the Question

The question given to students in this section is very straightforward and specific. Students are instructed to make a model for the water level given the data:

“Make a model for the water level on the given dates.” (Kendall Hunt, 2019).

The data is present, and the students have this somewhat singular instruction. The question is clear. There will need to be some assumptions made for variables as to how to structure the model based on the given data, but there is little ambiguity in the question and because of this, the authors have given it the highest lift score of two. On the CFM scale, this task is level six. This is not as high as the highest lift Viral Marketing prompt, which also has a lift score of two. This is because while the prompt has determined the context and students have to sort out what to do, the complete data is provided. There is little freedom here, the question has been posed specifically, and students are directed to make a very specific model.

QI-Quantities of Interest

In this prompt all of the key variables are declared, and the lift score is zero. In the excerpt below there is little question as to what quantities the student should use or how they apply (Kendall Hunt, 2019):

hours after 0:00 July 4	water level in feet
0	4.11
1	5.90
2	7.63
3	9.04

Figure 10: Swept Away Data Sample (Kendall Hunt, 2019)

In this case there is no question about the data on what the variables should be. The table has two titled columns. The teacher materials do allow for a scenario in which students determine their own data. In that scenario the overall lift level of the task would increase, but students would likely find data with similar time and water level variables. Even though they would need to parse through found data and make a decision, tide data will be given in depth and time format.

SD-Source of Data

In the given format the data is completely provided by the textbook. This is the reason for the low score on the lift analysis. While it has been stated that students could find their own source for the data, as it is written, the data is provided, and it is very applicable to the problem. Students will not have to parse through to find the meaning. A two-column table with units is all they need and all they are given.

AD-Amount of Data given

In this category the authors have rated this problem as a lift of one. According to their own matrix, this means that it is a medium lift, indicating that there may be extra data and students will have to decide which to use (Kendall Hunt, 2019). This is not true for the way the problem is presented initially. Students are given plenty of information to construct a model and they do not need to make any decision about which to use. It is not until students reach the third question where they check their model against new data:

"Ask your teacher for the high and low tide data for July 11, 2018. How accurately does your model predict these tides?" (Kendall Hunt, 2019).

While this data is provided, the teacher is to withhold it until students ask. This is an important aspect of the modeling process. The cyclical nature of modeling requires testing and revising the model (Bliss et al., 2019). Because students are instructed to ask for this data, the prompt meets the requirement of a medium lift.

M-The Model

The data provided here is sufficient for students to arrive at a trigonometric model of the tide cycles. Depending on the method students choose to arrive at their model or which specific data points they choose, the models will not be the same for all participants. Students have a lot of choice in how to represent their data to determine the type of model to use. It is in the construction of the model that this prompt reaches toward a higher level on California's spectrum, since students are able to choose a path to arrive at their model. The data does lend itself to a trigonometric model, but students may have success with polynomial models based on choices they make.

The first version of this task has an average lift score of one and is a level six on the CFM. This is because the teacher has provided a real-life question and context, but students still have freedom to choose representations of the data and a path to a model. Students will also assess the reasonableness of the model they develop (California Department of Education, 2013). Though this step is guided by the prompt itself, it is part of the modeling process as described in the GAIMME (2019) report.

> "Revise your model as needed. How accurately does your final model predict the high and low tides?" (Kendall Hunt, 2019).

The second version of the prompt provided students with more scaffolding designed to guide them to the model. The same data is provided, and the main goal is to create a model for the tides. Students still have to follow the modeling process and check their model against data to revise if necessary. The steps of creating the model, however, are provided in a more structured manner.

3.3.3 Swept Away Lower Lift:

The authors have provided the following lift analysis.

attribute	DQ	QI	SD	AD	M	avg
lift	1	0	0	1	1	0.6

Figure 11: Lift Analysis Swept Away (b) (Kendall Hunt, 2019)

This analysis only differs from the original in two categories. We will focus on those two categories to understand the differences.

DQ-Defining the Question

The lift score has decreased for this category from the original prompt. This can be attributed to the nature of the original question. The original prompt simply asks the student to make a model for the water level. In this case there are more leading questions about when high tides occur, and students are specifically asked to create a trigonometric model. This equates to a loss of choice. In a modeling situation, choice is a key component. The modeling in this version of the prompt is more guided and lower-level modeling. This difference can be seen in the following sequence of questions 1 through 4:

1. How many high tides are there in Boston each day? How many low tides? What is a good estimate for how often the high and low tides occur?

2. Estimate the average water level for July 4 and July 5, 2018.

3. Estimate how much the high and low tides differ from the average water level on July 4 and July 5, 2018.

4. Choose a trigonometric function (sine or cosine) to model the water level data. What horizontal shift will you need to model the data?

Figure 12: Swept Away Lower Lift Sequence (Kendall Hunt, 2019)

From this sequence it is clear that the questions are more guided. The four questions shown here occur before the students are asked to find a model. At that point they have already been instructed to use a trigonometric function as the base of the model and guided to find the horizontal shift. The assumptions have been made for the student as they go through this prompt, solidifying the lift score of one.

M-The Model

The model required for this version of the prompt is stated in the problem. This is another example of a loss of choice. Students that may need this scaffolding to get a model are missing that component of choice. The task specifically asks them to choose sine or cosine to create a model. Lowering the overall modeling efforts. Specifically, the lift score of one specifies that the modeler chooses the appropriate model from a list, and we have that situation here (Kendall Hunt, 2019).

3.3.4 CCSS

These two versions of this modeling prompt provide students a chance to demonstrate four Common Core State Standards. While students may not necessarily demonstrate these or demonstrate them to the same degree in every case, it is important to note their inclusion in the process.

- HSF-BF.B.3 "Identify the effect on the graph of replacing *f(x)* by *f(x)+k* , k*f(x)*, *f(kx)*, and *f(x) + k* for specific values of *k* (both positive and negative)" (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, Common Core State Standards for Mathematics).

In either prompt students are still instructed to make a model. Whether they arrive at a trigonometric model on their own, as is the expected result of the higher lift task, or they are instructed to make a trigonometric model specifically, they still have to determine which transformations of the parent function are required to fit it to the data. In the lower lift prompt, there is a fair amount of guidance. It specifically asks what type of horizontal shift is required to make the graph fit the data more accurately. One shortcoming of this type of problem, as far as using this modeling prompt as a way to test student acquisition of the skills associated with this standard, is that some students might use technology to create that model. In that case they miss this standard. The standard asks that students identify the effect of various operations on the graph of a function. If the data was fed into a computer and a function was output without students thinking of a parent function and how it may be transformed into one that fits the data, the purpose of the standards could be lost. There is no aspect of the modeling prompt that seeks to verify if students tweaked their model through an understanding of transformations. A teacher could easily rectify that through their instructions, expectations, or follow-up questions. If it was their desire to see students understanding of the transformations in this standard.

- HSF-IF.B.4 **“**For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship” (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, Common Core State Standards for Mathematics).

In order to demonstrate this standard, students need to be able to identify the high and low tides using their model and/or the data provided. Given the construction of either version, it seems reasonable that students would be able to demonstrate this standard. The standard assumes the students have a given function and that they merely interpret what they see in the graph/table and extract meaning from them. In this task students are asked to use the data in the table to create a model. In the higher lift task, they must notice that the table features a periodic nature with repeated maxima and minima. By analyzing this key feature, they are led to the conclusion that a trigonometric function is a good model. In the lower lift task, they are directed to find these patterns of maxima and minima and eventually build a trigonometric function. In this sense, they have interpreted those key features. While neither task explicitly asks for a sketch of the graph the descriptions high and low tides can be interpreted as “verbal descriptions” (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, Common Core State Standards for Mathematics) and then the standard is embedded well within the task.

> HSF-IF.C.7.E “Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude” (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, Common Core State Standards for Mathematics).

While a very thorough completion of this task might lead one to graph their model and make connections to how the period, midline, and amplitude show up and are related to the functions, it is not a required aspect of the prompt. Students will no doubt need knowledge of these aspects to create a quality model, but they can get away with bypassing a graph entirely. As there are no specific instructions to create a graphical representation of the table nor of the model once completed, students could simply use the data in the table, identifying key points from which to make their model. If the standard specifically asks that students graph, then there is no essential step in either version requiring a graphical representation.

> HSF-TF.B.5 “Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline” (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, Common Core State Standards for Mathematics).

If students have managed to create a final model, they will have undoubtedly demonstrated this standard. This standard is indeed the heart of this question. Students are provided periodic data with the goal of modeling it with trigonometric functions. In the lower lift problem, students are instructed to choose either sine or cosine to make their model. This choice can determine a little about what the students know of the functions. If the data and their model are to start at a high tide, then it would be more efficient to model with the cosine function.

3.3.5 Authenticity

The authenticity of this task lies in the initial setup the teacher has with students. Knowing what tides are and what causes them is an interesting scientific discussion. It is also one that needs to happen at ports and at beaches around the world. This discussion with students can enhance the authenticity of this problem. Tran and Dougherty tell us that an authentic question should be one that is actually posed in the real-world (2014). This is an important question that all coastal areas are concerned with. Tran and Dougherty also highlight that the information or data used should be accessible as well as the language of the problem (2014). This problem gives complete and clear data, and the language is direct and well within the expected high school level. The modeling task here is very authentic and fits as a good example of a task that is actually done all over the world and gives students a glimpse at how it could be accomplished.

3.3.6 Differences Between Versions

There are only two versions of this task, so there is inherently less differential and variation in the tasks. While the goal is the same, the second version is more of a walk-through of the process. The questions guide the student through all the steps necessary to begin testing out trigonometric functions that could fit the problem. Students are even told to choose between sine and cosine. While the first version has a lift score of one and the second has a lift score of 0.6, we can see the differences more clearly if we fit the task to the CMF. The first version is at level six. Students are given a version of a real-world task, but the teacher (or the prompt itself) has guided students with the specified data, assumptions, and the revision process (California

Department of Education, 2013). The second version falls to level four on California's spectrum due to the extra guided nature of the questions. The task is simplified and guarantees that students will arrive at the need for a sine or cosine equation to model the tides.

3.3.7 Summary

Each of the modeling tasks in the curriculum highlight different aspects of a true modeling situation. In this example the prompt leans into the actual model and the iterative process of refining and evaluating the model. Even in the more guided approach of the second version, students are still tasked with evaluating their model and refining it as needed. In the sense that these tasks are designed to help students become more comfortable with mathematical modeling, this prompt in particular shows students how to use collected data to make and refine a model. The first version is more authentic in the experience in modeling in that students have the necessary real data, but must tackle the problem and construct a model with less direction.

3.4 Modeling Prompt 3: Path of the Planets

The modeling prompt with the lowest lift and the least amount of authentic modeling is prompt three, The Path of The Planets. Students are tasked with developing a model that relates a planet’s orbital distance to the period of its orbit. Like the others, this prompt is differentiated into two versions. The first version has a lift analysis average of one and is similar to prompt seven, Swept Away. The second version has the lowest lift analysis average of all the modeling prompts. This would be an example of the author’s claim of the lowest lift modeling exercise. It showcases the lowest level of mathematical modeling available in the Algebra 2 modeling prompts.

In this task students are given a set of data for each of the eight planets’ distance from the sun and their orbital period. In both versions students are instructed to make a plot of the data, including which values to place on which axis. Students are then directly instructed to make a polynomial model for the data and to repeat this process by plotting the square root of all the data and creating another polynomial model. Students are asked to compare the two models. The task finished with a repetition of the task using data for the satellites of Jupiter. In the higher lift version, the students seek out this information on their own. In the lower lift, the data is provided. As the two versions of this task are so similar, differing only in the second section of the prompt in which students are to apply their model of the planets to the satellites of Jupiter. In the higher lift, the students are asked to seek this data out, and it is fully provided in the lower lift.

3.4.1 The Teacher's Role

The launch of this activity is simply to introduce the topic and sketch a picture of planets in orbit. Once there is an image of the solar system, teachers are to ask students to make a guess as to how long it takes each planet to orbit the sun. The final task in the setup is to directly tell students that there is a relationship between a planet's orbital distance and the orbital period. Once this is accomplished the students may begin the activity. Teachers should, as instructed by the curriculum, explicitly tell students that there is indeed a "relationship between a planet's distance from the sun and its orbital period" (Kendall Hunt, 2019). The materials also suggest to the teacher that this prompt would be well suited for the use of graphing technology and, specifically, technology that can create a polynomial model from a given table of values.

We will focus on the lower lift as an exemplar of what the authors consider their lowest level modeling situation.

3.4.2 Lowest Lift

The authors have given this prompt the following lift analysis.

attribute	DQ	QI	SD	AD	M	avg
lift	0	0	0	0	2	0.4

Figure 13: Lift Analysis Path of the Planets (Kendall Hunt, 2019)

DQ-Defining the Question

The authors have given this task a very light lift in terms of defining the question. This is evident in that the questions are descriptive and direct. Students are given tasks such as plotting the data and modeling it with a polynomial (Kendall Hunt, 2019). There is no question, only instructions to perform tasks. After the initial scenario is described, students are not given a chance to formulate a question. They are merely instructed to use the data.

planet	distance (millions of km)	period (days)
Mercury	57.9	88.0
Venus	108.2	224.7
Earth	149.6	365.2
Mars	227.9	687.0
Jupiter	778.6	4,331
Saturn	1,433.5	10,747
Uranus	2,872.5	30,589
Neptune	4,495.1	59,800

1. Plot the distance (x) and period (y) of each planet, and find a polynomial model that fits the data as well as possible. You may have to experiment with both the degree of your polynomial function and the number of terms.

Figure 14: Path of the Planets (Kendall Hunt, 2019)

This type of instruction gives little choice to students and provides no ambiguity. Students are able to apply modeling skills to the task but do not need to determine their own questions and it provides little opportunity for independent thoughts or approaches.

QI-Quantities of Interest

There is no ambiguity in the quantities of interest. The tasks are stated clearly and the data needed is declared overtly. Both versions of this task have the same direct nature leading students to seek the same data, which is provided. Students are even instructed as to which variable to place on which axis of their plot.

SD-Source of Data

All data required is presented in the lower lift version of the problem. The first half of the task asks students to use the provided data to plot distances and periods of planets. In the second half of the task, the students assigned the heavier lift are instructed to, “Look up the periods and distances of some [of the moons of Jupiter]” (Kendall hunt, 2019). In the lighter lift, data is provided for several of the moons. The lift score of zero demonstrates the explicit nature of the prompt.

AD-Amount of Data given

The amount of data given for the first half of this task is direct and not cluttered. The exact data is provided and apparent. Students do not need to peruse through any unneeded or extraneous data. In the second half, however, students working on the heavier lift need to browse information about the moons of Jupiter. For the lighter lift, the score is again zero, as there is no ambiguity or extra data provided. Students do not have to search, organize, or ignore any data.

M-The Model

Creating the actual model is that stage in which this task takes on its heaviest lift in either version of the task. In both cases students are instructed to take their data and make a mathematical model. They are specifically instructed to make a polynomial model. They then have to determine how accurate the model is. Students have data on the planet's orbit, and they can check their model against the provided data. In this category, making a mathematical model does require some choice. While they are instructed to make a polynomial model, they have to experiment and iterate in order to find the best fit. The prompt provides no guidance on how they attempt this. Students have assessed their model's accuracy and give the lift analysis its highest level and only non-zero score.

3.4.3 Authenticity of The Task

Even though this modeling prompt is a lower-level modeling situation, it does do well on a few of the authenticity criteria summarized by Tran and Dougherty (2014). The task is indeed representative of something that has been done. Johannes Kepler described the relationship between a planet's distance from the sun and its orbital period in the early 1600's and this fact is described directly in the student facing aspect of the modeling prompt (Kendall Hunt, 2019). This task is also clear on the data used and the purpose of completing the task. It uses authentic data as verified in the NASA Planetary Fact Sheet (NASA, 2018). It is also very directed; therefore, students can see each of the steps required to complete the task and find a model for this relationship and the authenticity of their purpose is evident.

3.4.4 CCSS

HSA-CED.A.2 "Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales" (National

> Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, Common Core State Standards for Mathematics).

The alignment to this standard is direct. Students are explicitly asked to plot the given data and to construct a model. They are directed to construct a polynomial model which indeed models the relationship between the distance and the orbital period. Students are even directed to place specific quantities on each axis so there is no question of which values are the independent or dependent variables. This task serves as a direct application and measure of this standard with little ambiguity. Questions arise in whether the standard is asking students to demonstrate this construction without computer aid. The modeling prompt instructs teachers to guide students to the use of such computerized aid.

> HSF-BF.A.1 “Write a function that describes a relationship between two quantities” (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, Common Core State Standards for Mathematics).

The description of this standard is similar to the previous standard. There is a slight nuance between writing an equation that models the relationship between two variables and writing a function that does the same. Students that successfully construct a polynomial model do indeed write a function that describes the relationship between the orbital distance and the period.

HSS-ID.B.6.A "Fit a function to the data; use functions fitted to data to solve problems in the context of the data. Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models" (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, Common Core State Standards for Mathematics).

In both versions of this prompt, students have to apply their model to a planet and verify that it matches the expected values. The standard expects students to be able to use their equation to solve problems in context. To the extent that the students have created the equation from the context in this situation, if they can apply it correctly and test their model, then they have met this standard. This standard emphasized using a given equation, and it would make more sense to test a student's ability to use a novel equation and the context of a situation to interpret their findings. If students have already constructed a model from the context, it would seem that their understanding of the relationships and context would be deep enough to apply it to a verification situation.

3.4.5 Summary

This problem does not fit well on the CMF since so much of it is guided and yet the construction of a mathematical model does require student choice and effort. This modeling task, while having a low level of modeling, still manages to maintain quality levels of authenticity and sets up good examples of most of the CCSS standards it claims to address. As described in the California Spectrum, we can consider this prompt a level four as the students are guided through the steps to a desired solution (California Department of Education, 2013). The students are given the data and the steps to create a formula to model that data. Students are also asked to assess their model via follow-up questions to make sure it is adequate.

One area where this prompt falls short of its intended purpose is in the details of the CCSS stated to be present. Even if a student completes each aspect of the prompt thoroughly, they may never address the idea that the model they have created is a function. Undoubtedly, the model they create is a function, but without acknowledging that relationship and being cognizant of the concept of functions built into the model, it is not representing HSF-BF.A.1, to the full extent. If there is to be a distinction between this standard and HSA-CED.A.2, which asks that students create equations in two or more variables, there would need to be cognition that the relation of those two variables is function.

Chapter 4 Discussion

In addition to the specific modeling prompts reviewed here, the IM Algebra 2 curriculum provides five other, individual modeling prompts with multiple versions at differing levels of mathematical modeling complexity. It also provides one modeling example. A hallmark of the modeling prompts provided is that they are carefully thought out with the process of mathematical modeling in mind. They focus on well-crafted structures, high quality questioning, detailed teacher guidance to ensure students follow the modeling process, and authentic context for students to explore. They also provide a wide variety in the levels of modeling, complexity, and they hit on many CCSS for Mathematics.

The teacher guidance included in the materials for the modeling prompts are both sufficiently thorough and brief. They do not include too much instruction for the teacher to overwhelm students or so much that it detracts from the modeling experience. A risk with a modeling task is that as students struggle, the teacher may be inclined to give more and more assistance. This limits the students' own mathematical thinking and reasoning. The teacher instructions provide specific warm-up or introduction activities and even include specific statements the teacher should make. This level of guidance allows teachers to activate student prior knowledge without giving away all the details of the task. As each of the modeling prompts is rated at a different level of difficulty, some prompts receive more teacher guidance than others, but the amount is brief and sufficient for students to begin exploring the data to complete the initial tasks of the prompt.

The modeling guidelines provided also give students an anchor to the modeling process. The charts and descriptions of the modeling process allow a teacher to continually reference the modeling process and the cyclical nature of its structure. The initial modeling prompt that serves as a template for how the subsequent prompts should be attempted is especially useful. A teacher taking advantage of this as an example of what students should be doing in their own process is setting the foundation for more complex independent student work. If there is a drawback to this initial modeling template it is that the problem itself relies heavily on algebra and equation manipulation. Students without as much confidence or experience with these manipulations may struggle to appreciate the overall modeling represented. However, if a teacher follows the overall guidance with fidelity, the students will be getting a good introduction to how mathematical modeling exists outside of the classroom.

The final part of the teacher's role in the modeling prompts is the choice the teacher has in assigning them. The various levels of modeling required for the prompts provide the teacher an opportunity to choose the appropriate level and hopefully build students' confidence to do more complex tasks throughout the course. While there is a specific order and time the IM curriculum suggests the prompts be assigned, the teacher that recognizes the strengths of their students can still differentiate and give the prompt with a more appropriate lift level.

The lift analysis provided by the authors helps a teacher determine which version of a prompt to assign or to help prepare for how much a class or student group may struggle. This lift analysis was determined to be largely accurate. The authors describe five categories and provide the lift for each. In every case studied here, the lift described was appropriate to the problem. Holistically, the lift for each problem largely matched the California Framework for Mathematical Modeling. A higher lift was associated with a higher level on that spectrum. The spectrum, however, fails to capture the same level of nuance as the authors' own lift analysis. The specificity involved in the lift analysis allows for the differentiation of tasks that may seem similar on the surface.

The IM curriculum also provides a set of CCSS that each prompt is thought to require to complete. Due to the open nature of modeling prompts and multiple pathways a student could take to reach a solution, even a correct one, the CCSS are not always fully represented. Modeling often lends itself to the use of computer aided mathematical regressions and graphing. In this sense, students using that technology may not be demonstrating their understanding of the concept described in the CCSS. In some cases, a model may be achieved for the students, and it may be accurate to a degree, but not what the prompt or teacher had in mind. One example would be a student misunderstanding of the periodic motion described in the prompt about tides. Students may incorrectly model the high and low tides over a shorter period and come up with a polynomial model. The model may do well in predicting the water levels, and with some modifications, students could use multiple polynomial models and make very accurate predictions. If this is the case, they would have completely bypassed the CCSS pertaining to an understanding of periodicity. It would take a skilled teacher to ensure the proper guidance to arrive at the specified standards in this case. Or, alternatively, the teacher may use that model to make connections to the relationship of sinusoidal functions and polynomials, and the strengths and drawbacks of each. They could even overlay the models and have students make these comparisons.

The open-ended nature of modeling tasks is one of their strengths and allows for a skilled teacher to help students on their journey of understanding. The inclusion of the CCSS for each prompt helps the teacher design their instruction and prepare for these off-target results. While it is difficult to rate an open-ended problem as to whether a student may or may not demonstrate the understanding the authors expect they will need to thoroughly respond to a prompt, the modeling in the IM Algebra 2 curriculum allows for freedom while initiating students into complex mathematical modeling situations.

4.1 Authenticity

One important aspect of modeling activities is authenticity. Tran and Doughtery (2014) tell us that authentic modeling tasks meet the specific criteria of event, question, purpose, information, language, and tools. The modeling tasks presented in IM present tasks with high levels of authenticity in most of these categories.

Of the modeling activities depicted here the levels of authenticity across the several categories are diverse, but each makes a strong case for being a problem that exists and is worth investigating. That is the case for many of the modeling situations throughout the IM Algebra 2 course. Not every aspect of authenticity, however, is well represented in every case by every modeling prompt.

Much of the difficulty in making an authentic prompt is in the category of the event. As Tran and Doughtry (2014) describe, this means making sure the modeling represents a real-life task or situation. While the IM prompts do a great job in structuring the tasks like one would in the real world, some are still contrived. An example of this is a prompt titled Exponential Situations (Kendall Hunt, 2019) which is based on exponential growth. The students are given a list of exponential growth situations and asked to choose one, or they are given the choice to construct a similar situation of their own making. The situations range from fantasy to scientific. This includes a situation about a vampire creating more vampires which, in turn, bite and create others. This situation is not a real-world example and loses authenticity in that sense, but it makes up for it in being an interesting question that may help students engage. It also meets other aspects of authenticity because in this fantastical event, real questions about how the population of vampires is going to grow would be very important to the citizens. The purpose of the task would also be clear and important. Gaining insight into the vampire epidemic is very important to the inhabitants of this unfortunate town. Other options in this prompt are more realistic involving bacterial growth and radioactive decay. Overall, the task is set up in a way to maintain authenticity while also giving students choice and keeping engagement.

Overall, the prompts use language appropriate to the high school level and they provide teacher guidance or explanations as necessary when they may be introducing terms they may be unfamiliar with. In all of the teacher materials, there is explicit guidance on giving students background information and preloading with the information they need to begin to access the modeling situation. Each situation is also simple enough for students to have reasonable experience with the concept. They know the planets orbit the sun, they know that companies have marketing departments, they have seen children's books, and they have at least heard about the tides. In each case the questions presented are natural things one might consider in a given situation. One example that fits this description but is perhaps the weakest connection is prompt number three. Students are presented with data on the planets' orbital period and their distance from the sun. It may not be natural for all students to make the connection and ask the question if there is a relationship between the two. Students may not even see that as an important question once they are led to ask it. There is not enough motivation to even ask the question. While the relationship is very interesting and provides a great example of modeling with equations and making predictions, getting there in an authentic way is a bit of a stretch. The problem get high marks in authenticity in other areas, however. The data presented is authentic, the situation is one that has actually been done, and the language used is appropriate and clear.

Chapter 5 Conclusions

The modeling prompts in the IM Algebra 2 provide students with opportunities to expand their knowledge. They allow students an opportunity to see mathematical thinking in action. While at the high school level the content is simplified enough to be accessible, the process of modeling and combining sources, as well as putting together a report, offers a sufficient analog to what real-world mathematics looks like. Students are learning the modeling process, and the IM prompts provide a balanced approach to high level modeling right down to an approachable lower level.

IM curates the various levels of modeling by varying different aspects of the modeling prompts. It provides teachers with an easy guide to see where the modeling prompt will be most challenging. The lift report for each modeling problem lets a teacher know where each student group may need the most support. These various levels also help students by changing the nature of the problem and their approach. Some mathematical problems may be easy to define, or easy to find the data, but maybe the question or the analysis is the challenging part. Some problems may be the exact opposite. The wide variety of modeling in the IM curriculum helps students round out each of those important mathematical qualities.

Along with the modeling structure, the prompts provide students with excellent opportunities to apply much of the mathematical content described in the CCSS for this level. Students can apply their skills in practical ways that require a deeper understanding of the concepts than a traditional cookie cutter type problem may allow. However, with the freedom ascribed to the modeler in the modeling process, not every standard may show up in the intended way. A skilled instructor is needed to guide students in the correct direction. The combinations of varied modeling structure and integrated mathematical concepts aligned with the CCSS makes the modeling prompts in IM excellent. The process of teaching modeling through these prompts improves the quality of the mathematical modeling present. With a thoughtful, skilled teacher at the helm, the IM Algebra 2 modeling prompts are both accessible modeling and a structure designed to teach students the process. Both high lift and lower lift opportunities show students the many aspects through largely authentic tasks.

Chapter 6 Suggestions

Many properties of the IM Modeling prompts described here are exemplary. Drawbacks include the level of skill a teacher needs to have in order to implement them properly. While IM attempts to address this with well-crafted teacher resources and guidance, it still takes some practice, classroom management, and high expectations to make these modeling situations function as intended. Solutions to this issue may be in high quality professional development and training opportunities focused on the modeling prompts specifically. As noted by Hans et al., through some concerted efforts and high-quality instructions, mathematical modeling can be taught to students (2009). With the amount of thought put into the analyzing each modeling prompt in each aspect of modeling, it is all meaningless if the teacher cannot convey the modeling cycle to the students. Students need to be actively engaged and working the steps. This can be one of the more difficult aspects for a teacher. Classroom management isn’t simply about keeping the students’ bad behaviors in check, it is about building a culture of learning in the classroom.

Teacher professional development opportunities should focus on the IM modeling prompts and the modeling process. This alone, however, will not be enough. Teachers should continue with training to develop a classroom that is conducive to the modeling process. It can be difficult to ask for a teacher to surrender the control of the classroom. Instead of instruction students in detail for every next step, the teacher’s role in a modeling activity is to guide students and help them reach their own conclusions, even if they are wrong the first time. Helping teachers gain confidence in this type of procedure may improve the modeling experience for both teachers and students.

In addition to training on the curriculum, especially in the areas of modeling and classroom management. I recommend that we support teaching programs that train highly qualified mathematics teachers in mathematics. Since modeling problems are complex and open-ended, it may take a teacher with a diverse and extensive mathematics background to fully engage with students on their path to a solution.

If the modeling process can be taught, a natural question arises as to what role it takes in a classroom. Many classrooms focus on results, scores, and grades. As noted here, there are some issues in using modeling prompts to determine proficiency at the skills in the CCSS. Indeed, modeling prompts are open ended and there may be multiple paths to a solution and multiple levels of correct solutions. How would a teacher score in such an open-ended process? A modeling task may not be the best format to gather information about student performance on the CCSS, but it does give a good picture of a student's mathematical mindset. The modeling prompts are not set up in the curriculum and summative assessments, and perhaps not even formative. They are not part of the sequenced curriculum and are intended as enrichment activities. These are teaching activities and can further a student’s education in mathematical and investigative techniques. This is in addition to strengthening their reasoning skills and persistence. The cyclical nature of the modeling process expects the first attempt is likely incorrect or incomplete. There is no rule stating that a path is or is not a dead end. Through the teacher’s guidance, the student should arrive at the appropriate conclusions, but it takes a skilled teacher, and this process is not well suited for assessment. Teachers should view these prompts as challenges, not only to the students, but to themselves.

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Appendix A: Spectrum of Mathematical Modeling

Spectrum of Mathematical Modeling (Examples Suitable for Upper Middle School and High School)
Level 9 (highest level): Students choose the context and the question. They experience the entire modeling process while confronting two or more iterations. The question may be practical or may concern something about which the student is curious.
Level 8: The context is provided by the teacher. Students determine a meaningful question related to the context and use the modeling process to determine an answer. *Example: Presented with a 12-pack of juice cans (or water bottles), what questions could be asked that would lead to a practical solution?*
Level 7: The teacher determines the context and poses the question to be answered. Students determine the relevant variables, make assumptions, and choose to simplify or ignore some of the variables. Students will need to justify their decision when making presentations. *Example: Find a better way to package juice cans.*
Level 6: Same as level 7, with the exception that the teacher guides students through the process of making assumptions and simplifications. Students develop and apply mathematical models and determine the reasonableness of the solutions. *Example: Find a better (more efficient) way to package juice cans. The discussion will determine that "efficient" means the ratio of the space used to the space available in the package. Students and teachers will assume the cans are perfect cylinders, restrict the package to the height of a single can, orient all cans in the same direction, and use a package that is a prism with congruent polygon bases (no shrink-wrap).*
Level 5: The teacher provides a simplified version of a real-life question and context. The problem is rich enough to allow for several solution paths and allow for access to various levels of mathematical background. *Example: Which package uses the highest percentage of space—a rectangular 12-pack, a triangular 10-pack, a trapezoidal 9-pack, or a hexagonal 7-pack? (All are prisms with a height equal to one can or bottle). Students make accurate representations of these packages and may determine the use of space through measurement, algebraic manipulation applied to polygons, or by using geometric sketchpad software.*
Level 4: Students are guided through the solution process that starts with a real-life context and question. The series of questions ensures that students will follow a particular path and use expected mathematics to solve the problem. The reasonableness of the solution is analyzed. *Example: Which is more efficient—the hexagonal 7-pack or the triangular 10-pack? Determine the percentage of space used in a hexagonal 7-pack.*
Level 3: A context and question are given. This is a real-world context with a mathematical focus. *Example: Six cans (circles) are placed together to form a triangle shape. The design engineer needs to find the height of the configuration. Determine the distance from the bottom can to the top can.*
Level 2: The context or real-world nature is incidental to the problem. The problem may even be contrived. *Example: Three circles are placed tangent to one another. Calculate the area bounded by the three circles.*
Level 1: There is no real-world context; the question is purely mathematical. *Example: Calculate the area of a circle with diameter equal to 2.5 inches.*

(California Department of Education, 2013)

Appendix B: Lift Analysis Chart

index	attribute	light lift (0)	medium lift (1)	heavy lift (2)
DQ	Defining the Question	well-posed question	elements of ambiguity; prompt might suggest ways assumptions could be made	freedom to specify and simplify the prompt; modeler must state assumptions
QI	Quantities of Interest	key variables are declared	key variables are suggested	key variables are not evident
SD	Source of Data	data is provided	modelers are told what measurements to take or data to look up	modelers must decide what measurements to take or data to look up
AD	Amount of Data given	modeler is given all the information they need and no more	some extra information is given and modeler must decide what is important; or, not enough information is given and modeler must ask for it before teacher provides it	modeler must sift through lots of given information and decide what is important; or, not enough information is given and modeler must make assumptions, look it up, or take measurements
M	The Model	a model is given in the form of a mathematical representation	type of model is suggested in words or by a familiar context; or, modeler chooses appropriate model from a provided list	careful thought about quantities and relationships or additional work (like constructing a scatterplot or drawing geometric examples) is required to identify type of model to use

(Kendall Hunt, 2019)

Appendix C: Advice on Modeling

Advice on Modeling

These are some steps that successful modelers often take, and questions that they ask themselves. You don't necessarily have to do all of these steps, or do them in order. Only do the parts that you think will help you make progress.

	**Understand the Question** Think about what the question means before you start making a strategy to answer it. Are there words you want to look up? Does the scenario make sense? Is there anything you want to get clearer on before you start? Ask your classmates or teacher if you need to.
	**Refine the Question** If necessary, rewrite the question you are trying to answer so that it is more specific.
	**Estimate a Reasonable Answer** If you don't have enough information to decide what's reasonable, try to come up with an answer that would be too low, and an answer that would be too high.
	**Identify Unknowns** • What are the meaningful quantities in this situation? Write them down. • What information would be useful to know? In order to get that information, you could: look it up, take a measurement, or make an assumption.
	**Gather Information** Write down any of the unknown information that you find. As you work, organize your information in a way that makes sense to you.

(Kendall Hunt, 2019)

<table>
<tr><td></td><td>Experiment!
Try different ideas to make progress toward answering your question. If you are stuck, think about:
Helpful ways to organize the information you have or organize your work
Questions you can answer using the information you have
Ways to represent mathematical relationships or sets of data (tables, equations, scatter plots, graphs, statistical plots)
Tools that are available for representing mathematics, both digital and analog</td></tr>
<tr><td></td><td>Check Your Reasoning
Do you have a first answer to your question? Great! See if it's reasonable.
Make sure you can explain what the answer means in terms of the original problem.
Check your precision: Is your answer overly precise (do you really need all those decimal places)? Not precise enough (were you overly aggressive with your rounding)?</td></tr>
<tr><td></td><td>Use and Improve Your Model
Did you make assumptions or measurements? How can you express your model more generally, so that it would work for a range of numbers instead of the specific numbers you used?
What are the limitations of your model? That is, what are some ways it is not realistic? Does it only work for certain inputs but not others? Are there any meaningful inputs affecting the outcome that are not accounted for? If possible, improve your model to take these into account.
What are the implications of your model? That is, what should people or organizations do differently or smarter as a result of what your model shows? What would be effective ways to communicate with them?
What are the areas for further research? That is, what new things are you wondering about that could be investigated, by you or someone else?</td></tr>
</table>

(Kendall Hunt, 2019)