One has

$$\sum_{\substack{i=1\\ i\in[j:\,Q(e_j)=0]}}^{t} |e_i| = \sum_{i=1}^{t} |e_i| - \sum_{\substack{i=1\\ i\in[j:\,Q(e_j)\neq 0]}}^{t} |e_i|. \qquad (5)$$

For some $r_1 \geq 0, r_2 \geq 0$, let the interval $[-r_1, r_2]$ be the dead zone of the quantizer. Then,

$$-r_1 < e_i < r_2 \text{ if and only if } i \in [j:\, Q(e_j) = 0]. \qquad (6)$$

(For a quantizer without a dead zone, $r_1 = r_2 = 0$.) Equations (5) and (6) imply that

$$\sum_{\substack{i=1\\ i\in[j:\,Q(e_j)\neq 0]}}^{t} |e_i| \geq \sum_{i=1}^{t} |e_i| - \max(r_1, r_2)t. \qquad (7)$$

Inserting (7) in (4), then

$$\lim_{t\to\infty} \frac{1}{t} \sum_{i=1}^{t} |e_i| \leq \max(r_1, r_2) + \frac{n\mu\beta^2 M^2}{2q}. \qquad (8)$$

Thus, (3.5) holds with $\delta = \max(r_1, r_2) + (n\mu\beta^2 M^2/2q)$. Equation (8) indicates that the mean absolute error of the QE algorithm increases with the width of the quantizer dead zone, as is intuitively expected. It is worth mentioning that (8) is satisfied by the example given in Section II. Indeed, as shown at the end of this example, $\lim_{t\to\infty} (1/t)\sum_{i=1}^{t} |e_i| = 1/8$. Due to (3) and (6), $r_1 = r_2 = 1/4$. Hence, (8) is satisfied.

References

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Reply to "Comments on 'A Comparison of Two Quantized State Adaptive Algorithms'"

W. A. Sethares and C. R. Johnson, Jr.

Dr. Eweda is correct in noting that our Theorem 1 does not apply to quantizers that incorporate a dead zone. However, the theorem, as stated in our paper, does not claim to apply to such quantizers.

Theorem 1 begins "Consider the Quantized Error algorithm (1.7)..." Equation (1.7) describes the quantized error algorithm, and the surrounding text explains the notation used in the algorithm. For instance, the regressor vector X_k, the prediction error e_k, and the quantizer $Q(\cdot)$) are all defined here and are used in the theorem and its proof. The first sentence after (1.7) says "In (1.7), $Q(\cdot)$ represents some quantization function: a bounded, discrete valued, element by element monotonic nondecreasing function *which does not change the sign of the argument*" (italics added). Quantization functions with dead zones, such as those in above comments, do change the sign of their arguments. To be specific, the *sign* function is usually defined to have three possible values: $+1$ when its argument is positive, 0 when its argument is zero, and -1 when its argument is negative. Dead zones typically map a region about 0 to 0, thus changing the sign from positive to zero, or from negative to zero. Thus, the quantization functions allowable in Theorem 1 do not include those with dead zones, contrary to the claims in the above comments.

In fact, dead zones are mentioned only once in our paper, inside the proof of Theorem 4, where we note that Theorem 4 also holds for dead zone quantizers. No such comment is made in or about Theorem 1. We welcome Dr. Eweda's result as an extension of our Theorem 1, but we deny that the theorem is in error.

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The Recursive Pyramid Algorithm for the Discrete Wavelet Transform

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***Abstract*—The recursive pyramid algorithm (RPA) is a reformulation of the classical pyramid algorithm (PA) for computing the discrete wavelet transform (DWT). The RPA computes the N-point DWT in real time (running DWT) using just $L(\log N - 1)$ words of storage, as compared with $O(N)$ words required by the PA. L is the length of the wavelet filter. The RPA is combined with the short-length FIR filter algorithms to reduce the number of multiplications and additions.**

I. Introduction

The discrete wavelet transform (DWT) can be computed in a efficient manner on general purpose computers using the pyramid algorithm (PA) developed by Mallat [1], [2]. There are a number of applications of the DWT where a "running" (or real time) implementation is desirable. Running implementations of the PA, for an N-point DWT, requires either $O(N)$ storage or $\log N$ filters cascaded together. Both these alternatives are expensive. We present a reformulation of the pyramid algorithm called the recursive pyramid algorithm (RPA). The RPA computes the N-point DWT in real time using just $L \log N - L$ cells of storage, where L is the length of the wavelet filter (QMF) and, generally, $L \ll N$. It computes the DWT in N steps (same as the PA) and the number of operations (multiplications+additions) required is comparable with that for the PA [3]. We show how the RPA can be computed using the short-length FIR algorithms described in [4]. This further reduces the number of operations required.

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