Patterns in the Pisano period and entry points of linear recurrence sequences modulo m

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Introduction

Theory

The aim of this project has been to generalize many well-known results regarding the Fibonacci sequence, $(F_n)_{n\geq 0}$, modulo m to other second-order linear recurrence sequences. This line of study began with the seminal work of D. D. Wall[5] in 1960. From Wall, we know $(F_n)_{n\geq 0}$ modulo m is periodic, that $e_F(m)$ is always defined, and that $e_F(m)$ divides $\pi_F(m)$. From the 1963 work of John Vinson,[4] we know that $\omega_F(m) = 1, 2$ or 4. In this research, we sought a precise characterization of $e_S(m)$, $\pi_S(m)$, and $\omega_S(m)$ in the Lucas, Pell, associated Pell, balancing, and Lucas-balancing sequences.

Applications

Beyond developing a full picture of particular sequence statistics, we also aimed to uncover their significance within the fundamental period. For example in $(F_n)_{n\geq 0}$ modulo 10, the "antipodal point" property holds.[1] That is, $F_n + F_{n+\frac{\pi_F(m)}{2}} \equiv 0 \pmod{10}$. We generalized this result beyond $m = 10$ and beyond the Fibonacci sequence. In our explorations of what information could be gleaned from $\omega_S(m)$, we found other interesting behaviors such as cyclic group structures and palindromes within the fundamental periods of particular sequences.

Tools

All data on relevant sequence statistics was generated with the use of Mathematica.

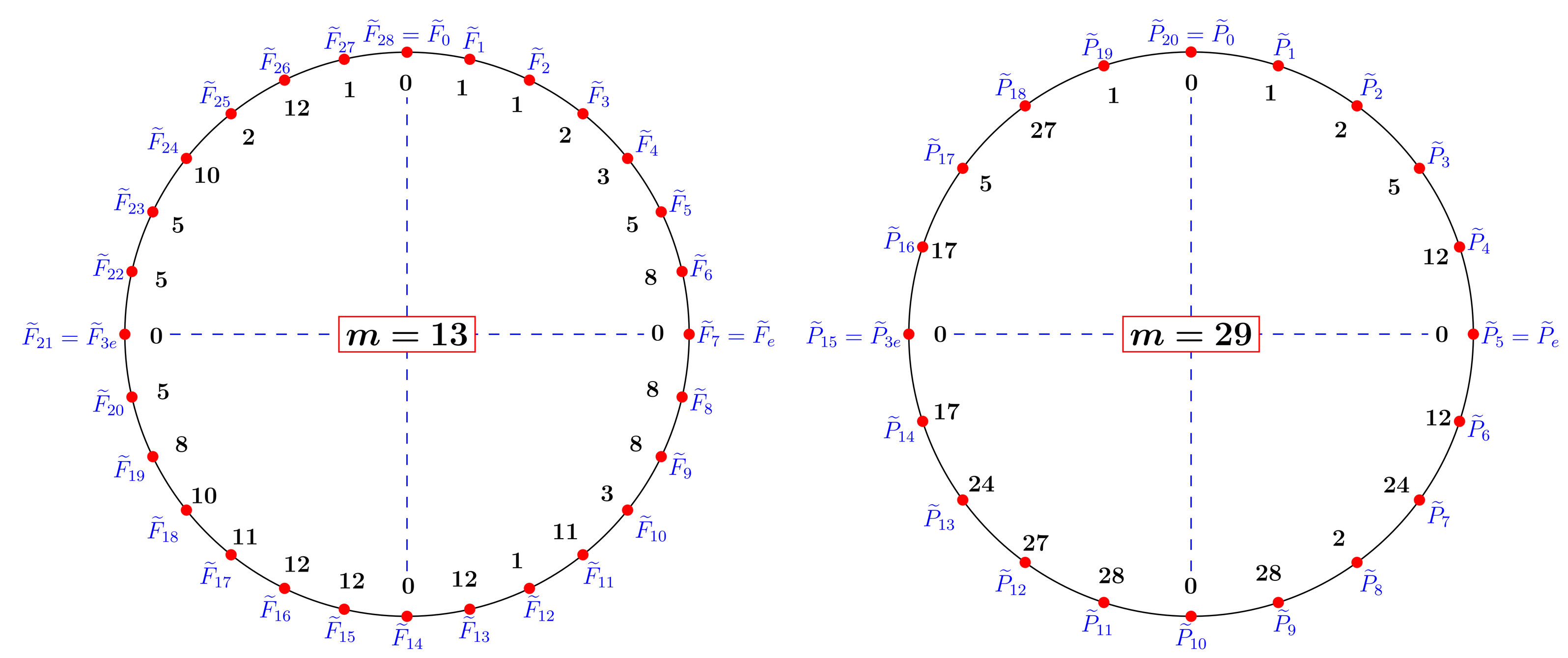
Definitions

- The **Fibonacci sequence** $(F_n)_{n\geq 0}$ is given by recurrence $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0$ and $F_1 = 1$. The **Lucas sequence** $(L_n)_{n\geq 0}$ is given by recurrence $L_n = L_{n-1} + L_{n-2}$ with $L_0 = 2$ and $L_1 = 1$.
- The **Pell sequence** $(P_n)_{n\geq 0}$ is given by recurrence $P_n = 2P_{n-1} + P_{n-2}$ with $P_0 = 0$ and $P_1 = 1$. The **associated Pell sequence** $(Q_n)_{n\geq 0}$ is given by recurrence $Q_n = 2Q_{n-1} + Q_{n-2}$ with $Q_0 = 1$ and $Q_1 = 1$.
- The **balancing sequence** $(B_n)_{n\geq 0}$ is given by recurrence $B_n = 6B_{n-1} - B_{n-2}$ with $B_0 = 0$ and $B_1 = 1$. The **Lucas-balancing sequence** $(C_n)_{n\geq 0}$ is given by recurrence $C_n = 6C_{n-1} - C_{n-2}$ with $C_0 = 1$ and $C_1 = 3$.
- Set $S_0 := a$ and $S_1 := b$ where $a, b \in \mathbb{N}$. The **Pisano period** of $(S_n)_{n\geq 0}$ modulo m is the smallest $r \in \mathbb{N}$ such that $S_r \equiv a \pmod{m}$ and $S_{r+1} \equiv b \pmod{m}$. Denote this value r by $\pi_S(m)$.
- The **entry point** of m in $(S_n)_{n\geq 0}$ is the smallest $r \in \mathbb{N}$ such that m divides S_r. Denote r by $e_S(m)$.
- Given $\pi_S(m)$ and $e_S(m)$, call the value $\frac{\pi_S(m)}{e_S(m)}$ the **order of m** denoted by $\omega_S(m)$.
- Let $\widetilde{S}_n$ denote the least residue class of $S_n \pmod{m}$. The sequence of $\widetilde{S}_n$ for $0 \leq n \leq r$, where $r = \pi_S(m)$ of $S_n \pmod{m}$ form the **fundamental period**.

Observations in the Fundamental Period

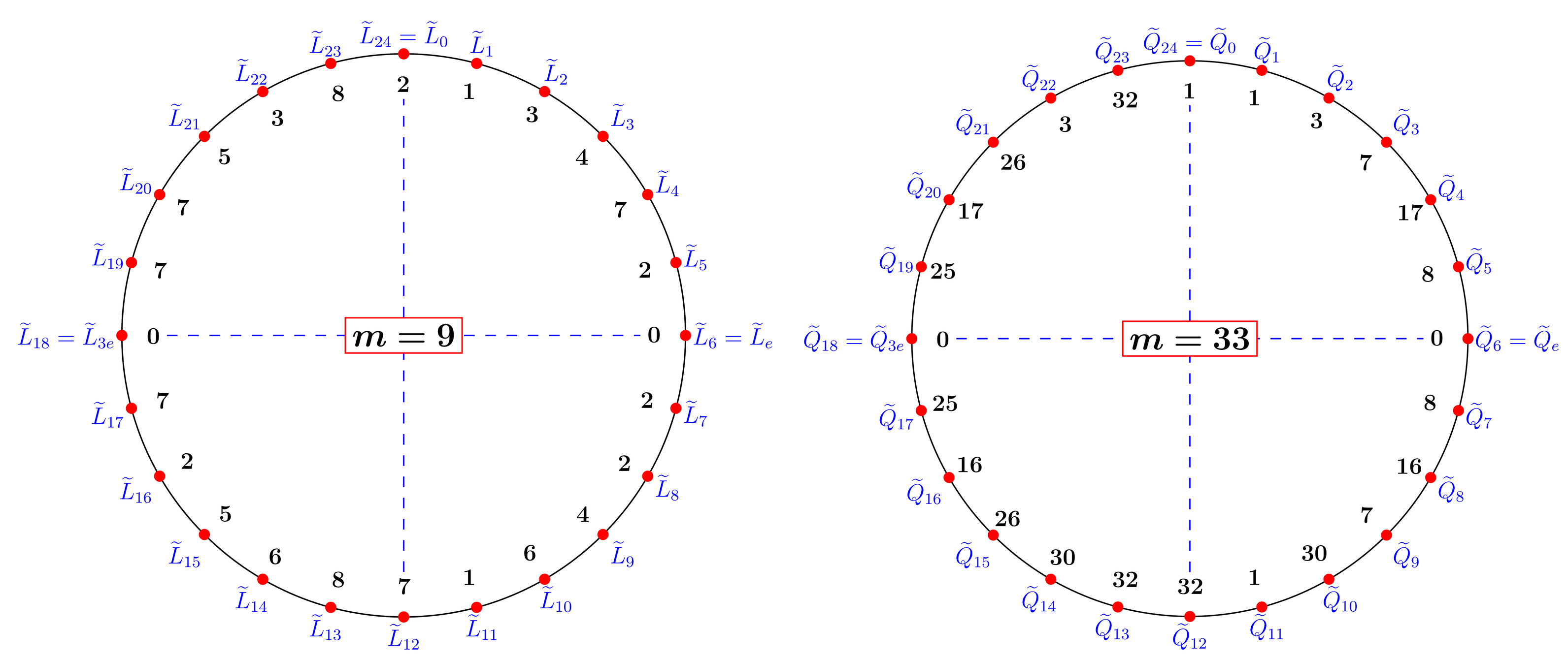
What does $\omega_S(m)$ tell us about the fudamental periods of our sequences modulo m?

Group theory results



Conjecture. Let $(S_n)_{n\geq 0}$ be $(F_n)_{n\geq 0}$ or $(P_n)_{n\geq 0}$. Consider m such that $\omega_S(m) = 4$, and set $e := e_S(m)$. Then, for $0 \leq k < 4$ we have $\widetilde{S}_{ek+1} \in U(m)$ where $U(m)$ is the set of units in the group $\mathbb{Z}_m$ and $\{\widetilde{S}_{ek+1}\}_{k=0}^{3}$ forms a cyclic group of order 4.

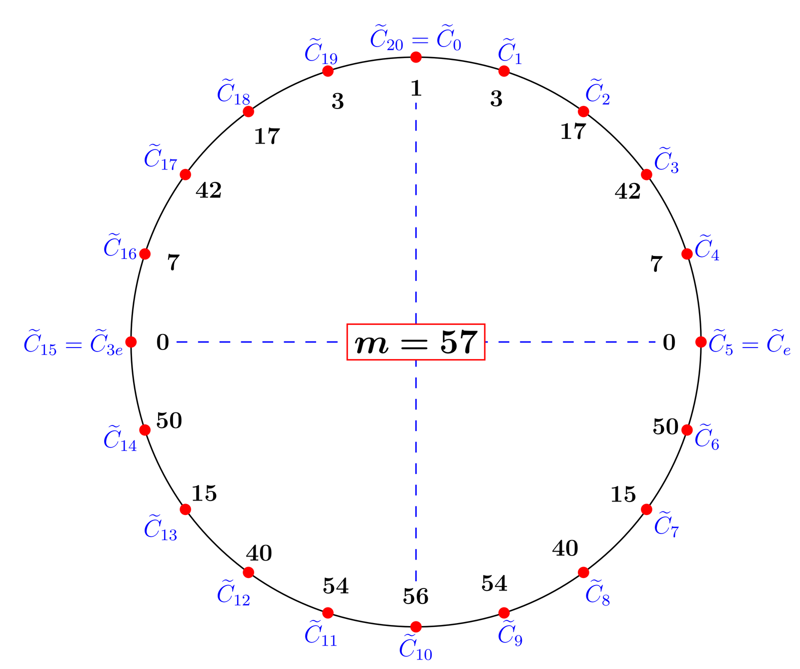
Antipodal points



Theorem. Let $m > 1$ and $(S_n)_{n\geq 0}$ be $(F_n)_{n\geq 0}$, $(L_n)_{n\geq 0}$, $(P_n)_{n\geq 0}$, $(Q_n)_{n\geq 0}$, or $(C_n)_{n\geq 0}$ and set $\widetilde{S}_n := S_n \pmod{m}$. If $\omega_S(m) = 4$, then for all $n \geq 0$, it follows that

$$\widetilde{S}_n + \widetilde{S}_{n+\frac{\pi_S(m)}{2}} \equiv 0 \pmod{m}.$$

Palindromes



Lemma. The vertices on the same horizontal slices of the fundamental circle equal each other. More precisely, we have $C_n \equiv C_{\pi-n} \pmod{m}$ for all $0 \leq n \leq \pi$.
Lemma. The vertices on the same vertical slices of the fundamental circle sum to the value m. More precisely, we have $C_{e+n} + C_{e-n} \equiv 0 \pmod{m}$ and $C_{3e+n} + C_{3e-n} \equiv 0 \pmod{m}$ for all $0 \leq n \leq e$
Theorem. Set $\pi := \pi_C(m)$. For all $m \geq 2$, the sequence $(\widetilde{C}_n)_{n=0}^{\pi}$ is a palindrome sequence. More precisely, for all $0 \leq n \leq \pi$, we have $C_n \equiv C_{\pi-n} \pmod{m}$.

Entry Points and Order

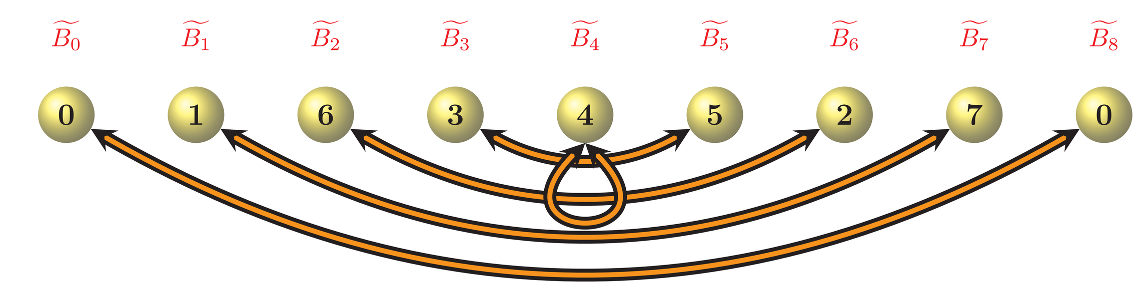
Results in the Fibonacci setting are well known. For $(P_n)_{n\geq 0}$ and $(B_n)_{n\geq 0}$, we used Renault[3] result $\omega_S(m)$ divides $2 \cdot ord_m(-b)$.

sequence	$e_S(m)$ always defined?	possible $\omega_S(m)$
$(F_n)_{n\geq 0}$	yes	1,2,4
$(L_n)_{n\geq 0}$	no	1,2,4
$(P_n)_{n\geq 0}$	yes	1,2,4
$(Q_n)_{n\geq 0}$	no	1,2,4
$(B_n)_{n\geq 0}$	yes	1,2
$(C_n)_{n\geq 0}$	no	1,4

Ongoing Work

Antipalindromes

We see both palindromes and antipalindromes in the balancing setting. When in the fundamental circle, we have $S_n + S_{\pi-n} \equiv 0 \pmod{m}$, we call the sequence an antipalindrome. **Question**: Can we predict when a particular modulus m will yield a palindrome versus an antipalindrome?



$(B_n)_{n\geq 0} \pmod{8}$ is an antipalidrome sequence.

Using order to predict values

In attempts to prove our group theory result, the question of whether we could predict the value of $\widetilde{S}_{e-1}$ was of particular interest. Renault[2] studied this topic and its relation to $\omega_S(m)$ in $(F_n)_{n\geq 0}$ modulo m. In $(P_n)_{n\geq 0}$ modulo m, we found that if $\omega_P(m) = 2$, then either $\widetilde{P}_{e-1}^2 \equiv 1$ or $-1 \pmod{m}$. The exception occurs when $m = 12$ or 15, similar to Fibonacci. Vinson[4] gives conditions for m which imply $\omega_F(m) = 2$, however they do not hold for Pell. **Questions**: What is the relationship between the m values such that $\omega_S(m) = 2$ in both $(F_n)_{n\geq 0}$ and $(P_n)_{n\geq 0}$ modulo m? What is the pattern amongst values of m such that $\omega_P(m) = 2$?

References and Acknowledgements

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