Technical Notes and Correspondence

Robust Stability of a Class of Polynomials with Coefficients Depending Multilinearly on Perturbations

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Abstract—**This paper gives necessary and sufficient conditions for robust stability of a family of polynomials $\mathcal{P}$ described as follows: each polynomial $P(s)$ in $\mathcal{P}$ is of the form $P(s) = U(s)V(s) + X(s)Y(s)$ where $U(s)$, $V(s)$, $X(s)$, and $Y(s)$ are interval polynomials.**

I. Introduction and Problem Formulation

The seminal theorem of Kharitonov [1] has spurred a large number of papers dealing with special cases and variations of the following robustness problem.

Given: A region $\mathfrak{D}$ in the complex plane and a family of polynomials $\mathcal{P}$. Each polynomial in this family is described by its vector of coefficients $p \in \boldsymbol{R}^{n+1}$ which is known only within given bounds $\boldsymbol{P} \subset \boldsymbol{R}^{n+1}$.

Problem: Provide some "computationally tractable" scheme to determine if every polynomial in $\mathcal{P}$ has all its zeros in the interior of $\mathfrak{D}$. In such a case, the family $\mathcal{P}$ is said to be *robustly* $\mathfrak{D}$*-stable* and for the special case when $\mathfrak{D}$ is the left half plane, $\mathcal{P}$ is said to be *robustly stable*.

Motivation for current research is derived from the fact that polynomials associated with a control system typically have a *coefficient set* $\boldsymbol{P}$ which can be quite "ugly" because the physical parameters might enter into the coefficient vector $\boldsymbol{p}$ in a rather complicated manner. In the work of Kharitonov, $\boldsymbol{P}$ is a rectangle and for the case when $\boldsymbol{P}$ is a polytope, one highlight is the Edge Theorem of Bartlett *et al.* [2] (see also [3], [4] for a more lengthy discussion of the literature).

This note is concerned with the case when $\boldsymbol{P}$ is obtained via a so-called multilinear perturbation structure. In this regard, the takeoff point for a number of papers is the Mapping Theorem (see [5]) (see also [6]–[9]). In the cases of [6] and [9], a degree of conservatism is present due to overbounding of $\boldsymbol{P}$, whereas in [7] and [8], an iterative procedure (involving cutting hyperplanes) leads to better approximations at the expense of computational complexity. Other results involving rather restrictive assumptions on the multilinearity are given in [10]–[12].

This note also involves restrictions on the multilinearity but, in contrast to existing literature, these restrictions are derived from physical considerations stemming from analysis of a closed-loop "interval" feedback system. More specifically, a cascade connection of SISO transfer functions leads us to consider a polynomial of the form

$$P(s) = U(s)V(s) + X(s)Y(s)$$

where $U(s)$, $V(s)$, $X(s)$, and $Y(s)$ are interval polynomials with coefficient vectors belonging to *bounding rectangles* $\boldsymbol{U} \subset \boldsymbol{R}^{n_u+1}$, $\boldsymbol{V} \subset \boldsymbol{R}^{n_v+1}$, $\boldsymbol{X} \subset \boldsymbol{R}^{n_x+1}$, $\boldsymbol{Y} \subset \boldsymbol{R}^{n_y+1}$, respectively. Hence, we obtain four interval polynomial families denoted by $\mathcal{U}$, $\mathcal{V}$, $\mathcal{X}$, and $\mathcal{Y}$.

This note provides an easily verifiable necessary and sufficient condition for robust stability of $\mathcal{P}$. The main result indicates that all polynomials in $\mathcal{P}$ have their zeros in the strict left half plane if and only if two requirements are satisfied at each frequency $w \in \boldsymbol{R}$. The first requirement is a zero exclusion condition involving "four Kharitonov rectangles" associated with U, V, X, and Y. The second requirement is that a specially constructed θ*-parameterized* set of 16 intervals $I_1(w, \theta), I_2(w, \theta), \cdots, I_{16}(w, \theta)$ must cover the positive reals for each $\theta \in [0, 2\pi]$.

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II. Notation and Assumption

In this section, we develop some notational preliminaries which will be useful for describing the geometry underlying the results. We also give an assumption which remains in force throughout this note.

Notation 2.1: Consider an interval polynomial family $\mathcal{W}$ described by

$$W(s) = w_0 + w_1 s + w_2 s^2 + \cdots + w_{n_w} s^{n_w}$$

where $w_i \in [w_i^-, w_i^+]$ for $i = 0, 1, \cdots, n_w$ and denote the four Kharitonov polynomials associated with $W(s)$ by

$$K_1^W(s) = w_0^- + w_1^- s + w_2^+ s^2 + w_3^+ s^3 + w_4^- s^4 + w_5^- s^5 + w_6^+ s^6 + \cdots;$$

$$K_2^W(s) = w_0^+ + w_1^+ s + w_2^- s^2 + w_3^- s^3 + w_4^+ s^4 + w_5^+ s^5 + w_6^- s^6 + \cdots;$$

$$K_3^W(s) = w_0^+ + w_1^- s + w_2^- s^2 + w_3^+ s^3 + w_4^+ s^4 + w_5^- s^5 + w_6^- s^6 + \cdots;$$

$$K_4^W(s) = w_0^- + w_1^+ s + w_2^+ s^2 + w_3^- s^3 + w_4^- s^4 + w_5^+ s^5 + w_6^+ s^6 + \cdots.$$

For fixed frequency $w \in \boldsymbol{R}$, the *cross section* of $\mathcal{W}$ at w is given by

$$\mathcal{W}_w \doteq \{W(jw) : W(s) \in \mathcal{W}\}.$$

Since $\mathcal{W}$ is an interval polynomial family, it is well known that this cross section $\mathcal{W}_w$ is a rectangle in complex plane $\mathcal{C}$ defined by the $K_i^W(s)$ above (e.g., see [13], [14]).

Next, given a complex number $z = re^{j\theta}$, let

$$z\mathcal{W}_w \doteq \{zW(jw) : W(s) \in \mathcal{W}\}.$$

Notice that $z\mathcal{W}_w$ is still a rectangle—obtained by magnifying all points in $\mathcal{W}_w$ by r and rotation through an angle θ. Having this notation in hand, we define the sets

$$\Omega_{UX}(w) \doteq \{z \in \mathcal{C} : U_0(jw) = zX_0(jw) \text{ for some } U_0(s) \in \mathcal{U}, X_0(s) \in \mathcal{X}\}$$

and

$$\Omega_{YV}(w) \doteq \{z \in \mathcal{C} : Y_0(jw) = -zV_0(jw) \text{ for some } Y_0(s) \in \mathcal{Y}, V_0(s) \in \mathcal{V}\}.$$

Finally, given two complex numbers z_1 and z_2, we use the inner product notation

$$\langle z_1, z_2 \rangle \doteq \operatorname{Re} z_1 \cdot \operatorname{Re} z_2 + \operatorname{Im} z_1 \cdot \operatorname{Im} z_2.$$

To avoid trivialities, we impose the following assumption which can be checked *a priori* using the Routh–Hurwitz criteria.

Assumption 2.2: There exists at least one strictly stable polynomial $P_*(s) \in \mathcal{P}$.

III. A Sequence of Preliminary Lemmas

In this section, we provide a sequence of preliminary lemmas. The reader who is interested solely in the final result can proceed directly to Section IV.

Lemma 3.1: The following condition is necessary for robust stability of $\mathcal{P}$: for each $w \in \boldsymbol{R}$

$$0 \notin (\mathcal{U}_w \cup \mathcal{V}_w) \cap (\mathcal{X}_w \cup \mathcal{Y}_w).$$

Proof: Proceeding by contradiction, suppose $0 \in (\mathcal{U}_{w_0} \cup \mathcal{V}_{w_0}) \cap (\mathcal{X}_{w_0} \cup \mathcal{Y}_{w_0})$ for some $w_0 \in \boldsymbol{R}$. It follows that there exist some $U_0(s) \in \mathcal{U}$, $V_0(s) \in \mathcal{V}$, $X_0(s) \in \mathcal{X}$, and $Y_0(s) \in \mathcal{Y}$ such that $U_0(jw_0) = 0$ or $V_0(jw_0) = 0$, and $X_0(jw_0) = 0$ or $Y_0(jw_0) = 0$. Hence,

$$P_0(jw_0) \doteq U_0(jw_0)V_0(jw_0) + X_0(jw_0)Y_0(jw_0) = 0$$

which implies that $s = jw_0$ is a zero of $P_0(s)$. This contradicts the robust stability of $\mathcal{P}$. □

Lemma 3.2: $\mathcal{P}$ is robustly stable if and only if the following two conditions are satisfied.

Condition 3.2.1: For each $w \in \mathbf{R}$, $0 \notin (\mathcal{U}_w \cup \mathcal{V}_w) \cap (\mathcal{X}_w \cup \mathcal{Y}_w)$.

Condition 3.2.2: For each $w \in \mathbf{R}$, $\Omega_{UX}(w) \cap \Omega_{YV}(w) = \emptyset$.

Proof: Since Lemma 3.1 establishes that Condition 3.2.1 is necessary, we assume Condition 3.2.1 holds and need only prove that $\mathcal{P}$ is robustly stable if and only if Condition 3.2.2 holds.

Necessity: Proceeding by contradiction, suppose $\Omega_{UX}(w_0) \cap \Omega_{YV}(w_0) \neq \emptyset$ for some $w_0 \in \mathbf{R}$. Then there exists a complex number $z_0 \in \mathcal{C}$, and polynomials $U_0(s) \in \mathcal{U}$, $V_0(s) \in \mathcal{V}$, $X_0(s) \in \mathcal{X}$, and $Y_0(s) \in \mathcal{Y}$ such that $U_0(jw_0) = z_0 X_0(jw_0)$ and $Y_0(jw_0) = -z_0 V_0(jw_0)$. Taking $P_0(s) \doteq U_0(s)V_0(s) + X_0(s)Y_0(s)$, it follows that $P_0(jw_0) = 0$. Hence, $s = jw_0$ is a zero of $P_0(s)$ which contradicts the robust stability of $\mathcal{P}$.

Sufficiency: Proceeding by contradiction, suppose that $\mathcal{P}$ is not robustly stable. Then it follows that there exists an unstable polynomial in $\mathcal{P}$, denoted as $P_1(s) = U_1(s)V_1(s) + X_1(s)Y_1(s)$ where $U_1(s) \in \mathcal{U}$, $V_1(s) \in \mathcal{V}$, $X_1(s) \in \mathcal{X}$, and $Y_1(s) \in \mathcal{Y}$. By Assumption 2.2, there also exists a strictly stable polynomial in $\mathcal{P}$, denoted as $P_*(s) = U_*(s)V_*(s) + X_*(s)Y_*(s)$ where $U_*(s) \in \mathcal{U}$, $V_*(s) \in \mathcal{V}$, $X_*(s) \in \mathcal{X}$, and $Y_*(s) \in \mathcal{Y}$. Now, for $\lambda \in [0, 1]$, consider the family of polynomials

$$P(\lambda, s) = [\lambda U_1(s) + (1-\lambda)U_*(s)][\lambda V_1(s) + (1-\lambda)V_*(s)] + [\lambda X_1(s) + (1-\lambda)X_*(s)][\lambda Y_1(s) + (1-\lambda)Y_*(s)].$$

It is clear that $P(\lambda, s) \in \mathcal{P}$, $P(0, s) = P_*(s)$, and $P(1, s) = P_1(s)$. Using continuity of the zeros of a polynomial with respect to its coefficients, it follows that $P(\lambda^*, jw_0) = 0$ for some $\lambda^* \in (0, 1]$ and some $w_0 \in \mathbf{R}$. Namely, there exist $U_0(s) \in \mathcal{U}$, $V_0(s) \in \mathcal{V}$, $X_0(s) \in \mathcal{X}$, and $Y_0(s) \in \mathcal{Y}$ such that $U_0(jw_0)V_0(jw_0) + X_0(jw_0)Y_0(jw_0) = 0$ which holds only if $U_0(jw_0) \neq 0$, $V_0(jw_0) \neq 0$, $X_0(jw_0) \neq 0$, and $Y_0(jw_0) \neq 0$ in light of Condition 3.2.1. Then taking

$$z_0 \doteq \frac{U_0(jw_0)}{X_0(jw_0)} \doteq -\frac{Y_0(jw_0)}{V_0(jw_0)}$$

we have $z_0 \in \Omega_{UX}(w_0) \cap \Omega_{YV}(w_0)$ which implies that $\Omega_{UX}(w_0) \cap \Omega_{YV}(w_0) \neq 0$. This contradicts Condition 3.2.2. □

Lemma 3.3: Let $w \in \mathbf{R}$ and $z \in \mathcal{C}$ be fixed. Then $z \in \Omega_{UX}(w)$ if and only if $\mathcal{U}_w \cap [z\mathcal{X}_w] \neq \emptyset$. Similarly, $z \in \Omega_{YV}(w)$ if and only if $\mathcal{Y}_w \cap [-z\mathcal{V}_w] \neq \emptyset$.

Proof: Immediate from the definitions of the sets Ω_{UX} and Ω_{YV}. □

Lemma 3.4: Let $w \in \mathbf{R}$ and $z = re^{j\theta}$ with $r > 0$ and $\theta \in [0, 2\pi]$ be fixed. Then $z \in \Omega_{UX}(w)$ if and only if the following conditions hold.

Condition 3.4.1: $r\langle K_i^X(jw), e^{-j\theta}\rangle \geq \operatorname{Re} K_1^U(jw)$ for some $i \in \{1, 2, 3, 4\}$.

Condition 3.4.2: $r\langle K_i^X(jw), e^{-j\theta}\rangle \leq \operatorname{Re} K_2^U(jw)$ for some $i \in \{1, 2, 3, 4\}$.

Condition 3.4.3: $r\langle K_i^X(jw), e^{j\left(\frac{\pi}{2}-\theta\right)}\rangle \geq \operatorname{Im} K_3^U(jw)$ for some $i \in \{1, 2, 3, 4\}$.

Condition 3.4.4: $r\langle K_i^X(jw), e^{j\left(\frac{\pi}{2}-\theta\right)}\rangle \leq \operatorname{Im} K_4^U(jw)$ for some $i \in \{1, 2, 3, 4\}$.

Condition 3.4.5: $\langle K_i^U(jw), e^{j\theta}\rangle \geq r \operatorname{Re} K_1^X(jw)$ for some $i \in \{1, 2, 3, 4\}$.

Condition 3.4.6: $\langle K_i^U(jw), e^{j\theta}\rangle \leq r \operatorname{Re} K_2^X(jw)$ for some $i \in \{1, 2, 3, 4\}$.

Condition 3.4.7: $\langle K_i^U(jw), e^{j\left(\frac{\pi}{2}+\theta\right)}\rangle \geq r \operatorname{Im} K_3^X(jw)$ for some $i \in \{1, 2, 3, 4\}$.

Condition 3.4.8: $\langle K_i^U(jw), e^{j\left(\frac{\pi}{2}+\theta\right)}\rangle \leq r \operatorname{Im} K_4^X(jw)$ for some $i \in \{1, 2, 3, 4\}$.

Proof:

Necessity: Proceeding by contradiction, if Condition 3.4.1 fails, this means that $r\langle K_i^X(jw), e^{-j\theta}\rangle < \operatorname{Re} K_1^U(jw)$ for $i = 1, 2, 3, 4$. That is, the rotated rectangle $z\mathcal{X}_w$ lies to the left of the rectangle $\mathcal{U}_w$. Hence, $z \notin \Omega_{UX}(w)$. Similarly, we can consider Conditions 3.4.2–3.4.4. If Condition 3.4.5 fails, this means that $\langle K_i^U(jw), e^{j\theta}\rangle < r \operatorname{Re} K_i^X(jw)$ for $i = 1, 2, 3, 4$. This is equivalent to having the rotated rectangle $(1/z)\mathcal{U}_w$ lie to the left of the rectangle $\mathcal{X}_w$. It follows that $z \notin \Omega_{UX}(w)$. Similarly, we can consider Conditions 3.4.6–3.4.8.

Sufficiency: Notice that $\mathcal{U}_w \cap [z\mathcal{X}_w] = \emptyset$ if and only if either all points in $\mathcal{U}_w$ are in one side of $z\mathcal{X}_w$ or all points in $z\mathcal{X}_w$ are in one side of $\mathcal{U}_w$. Based on this fact, we get the proof. □

We now continue with the problem of characterizing the sets $\Omega_{UX}(w)$ and $\Omega_{YV}(w)$. Motivated by Lemma 3.4, we will define eight intervals $I_1(\theta, w), I_2(\theta, w), \cdots, I_8(\theta, w)$ associated with $\Omega_{UX}(w)$ and another eight intervals $I_9(\theta, w), I_{10}(\theta, w), \cdots, I_{16}(\theta, w)$ associated with $\Omega_{YV}(w)$. Note that these 16 intervals will play an important role in the main result. We describe the construction of $I_1(\theta, w)$ below and simply list the nearly identical formulas for $I_2(\theta, w)$ through $I_{16}(\theta, w)$ in the Appendix.

Construction of $I_1(\theta, w)$: For fixed $i \in \{1, 2, 3, 4\}$, we define a lower bound function

$$R_{1i}^-(\theta, w) \doteq \begin{cases} \max\left\{\dfrac{\operatorname{Re} K_1^U(jw)}{\langle K_i^X(jw), e^{-j\theta}\rangle}, 0\right\} \\ \quad \text{if } \left\langle K_i^X(jw), e^{-j\theta}\right\rangle < 0; \\ 0 \quad \text{if } \left\langle K_i^X(jw), e^{-j\theta}\right\rangle \geq 0 \end{cases}$$

and an upper bound function

$$R_{1i}^+(\theta, w) \doteq \begin{cases} \max\left\{\dfrac{\operatorname{Re} K_1^U(jw)}{\langle K_i^X(jw), e^{-j\theta}\rangle}, 0\right\} \\ \quad \text{if } \left\langle K_i^X(jw), e^{-j\theta}\right\rangle > 0; \\ 0 \quad \text{if } \left\langle K_i^X(jw), e^{-j\theta}\right\rangle = 0 \text{ and } \operatorname{Re} K_1^U(jw) \leq 0; \\ +\infty \quad \text{otherwise.} \end{cases}$$

To complete the construction of $I_1(\theta, w)$, let

$$R_1^-(\theta, w) \doteq \max_{1 \leq i \leq 4} R_{1i}^-(\theta, w)$$

and

$$R_1^+(\theta, w) \doteq \min_{1 \leq i \leq 4} R_{1i}^+(\theta, w).$$

Then define

$$I_1(\theta, w) \doteq (R_1^-(\theta, w), R_1^+(\theta, w)).$$

Lemma 3.5: Let $w \in \mathbf{R}$ and $z = re^{j\theta}$ with $r > 0$ and $\theta \in [0, 2\pi]$ be fixed. Then $z \in \Omega_{UX}(w)$ if and only if $r \notin I_k(\theta, w)$ for $k = 1, 2, 3, 4, 5, 6, 7, 8$.

Proof: We work with Condition 3.4.1, and note that a nearly identical analysis is used for Condition 3.4.k for $k = 2, 3, 4, 5, 6, 7, 8$. Indeed, for fixed $w \in \mathbf{R}$, notice that Condition 3.4.1 holds if and only if for some $i \in \{1, 2, 3, 4\}$, one of the following three conditions is satisfied.

Condition 3.5.1: $r \geq \operatorname{Re} K_1^U(jw)/\langle K_i^X(jw), e^{-j\theta}\rangle$, if $\langle K_i^X(jw), e^{-j\theta}\rangle > 0$.

Condition 3.5.2: $r \leq \operatorname{Re} K_1^U(jw)/\langle K_i^X(jw), e^{-j\theta}\rangle$, if $\langle K_i^X(jw), e^{-j\theta}\rangle < 0$.

Condition 3.5.3: $\operatorname{Re} K_1^U(jw) \leq 0$, if $\langle K_i^X(jw), e^{-j\theta}\rangle = 0$.

It is clear that Condition 3.5.1 or Condition 3.5.3 hold if and only if $r \geq R_{1i}^+(\theta, w)$ and Condition 3.5.2 holds if and only if $r \leq R_{1i}^-(\theta, w)$. Then Condition 3.4.1 holds if and only if $r \notin I_1(\theta, w)$. □

In the same way, we can characterize $\Omega_{YV}(w)$ using $I_9(\theta, w)$ through $I_{16}(\theta, w)$.

IV. Main Result

In this section, we provide the main result.

Theorem 4.1: $\mathcal{P}$ is robustly stable if and only if the following two conditions are satisfied.

Condition 4.1.1: For each $w \in \mathcal{R}$, $0 \notin (\mathcal{U}_w \cup \mathcal{V}_w) \cap (\mathcal{X}_w \cup \mathcal{Y}_w)$.

Condition 4.1.2: For each $w \in \mathcal{R}$, and each $\theta \in [0, 2\pi]$, the 16 intervals $I_1(\theta, w), I_2(\theta, w), \cdots, I_{16}(\theta, w)$ cover the set of positive reals, i.e., $\cup_{i=1}^{16} I_i(\theta, w) = (0, +\infty)$.

Proof: In view of the necessity of Condition 4.1.1 (recall Lemma 3.1), the conclusion of the theorem is guaranteed if and only if Condition 4.1.2 holds. Indeed, by Lemma 3.2, robust stability of $\mathcal{P}$ is guaranteed if and only if $\Omega_{UX}(w) \cap \Omega_{YV}(w) = \emptyset$ for all $w \in R$. Using Lemma 3.3, it follows that for each nonzero complex number z and each $w \in R$, either

$$\mathcal{U}_w \cap z\mathcal{X}_w = \emptyset \tag{1}$$

or

$$\mathcal{Y}_w \cap [-z\mathcal{V}_w] = \emptyset. \tag{2}$$

Writing $z = re^{\theta}$ with $r > 0$ and $\theta \in [0, 2\pi]$, Lemma 3.4 indicates that (1) holds if and only if

$$r \in \bigcup_{k=1}^{8} I_k(\theta, w). \tag{3}$$

In the same way, (2) holds if and only if

$$r \in \bigcup_{k=9}^{16} I_k(\theta, w). \tag{4}$$

Combining (3) and (4), we obtain covering Condition 4.1.2. □

Conditions 4.1.1 and 4.1.2 are both easy to check because we have specific formulas for the endpoints of the intervals $I_i(\theta, w)$ and the vertices of the Kharitonov rectangles. For computational purposes, it is also important to point out that Conditions 4.1.1 and 4.1.2 will always be satisfied at sufficiently high frequency.

V. Numerical Example

To illustrate the application of Theorem 4.1, we consider a cascade connection of two blocks $U(s)/X(s)$ and $V(s)/Y(s)$ with unity feedback. We take $U(s) = 3s + 2$, $X(s) = s^2 - 3s + 10$, $V(s) = 20s + 23$, and $Y(s) = s^2 + 10s + 5$ and view $U(s)/X(s)$ as *the uncompensated system* and $V(s)/Y(s)$ as *actuator dynamics*. Analyzing the closed-loop stability of this system involves studying the zeros of the polynomial $P(s) = U(s)V(s) + X(s)Y(s)$. Before applying the result of this note, we develop two benchmarks.

Benchmark 1—The Perturbation Free System: Notice that the uncompensated system $U(s)/X(s)$ has denominator $X(s) = s^2 - 3s + 10$ which is unstable. However, with the inclusion of $V(s)/Y(s)$ and the feedback, the closed-loop polynomial becomes $P(s) = s^4 + 7s^3 + 45s^2 + 194s + 96$ which is easily verified to be strictly stable.

Benchmark 2—Perturbations only in U(s) and X(s): Now we consider the case when $U(s)/X(s)$ has perturbations in its coefficients and $V(s)$ and $Y(s)$ are fixed. That is, we consider $U(s) = [3+u_1]s+[2+u_0]$; $|u_0| \le 0.3$; $|u_1| \le 0.3$ and $X(s) = s^3 - [3+x_1]s + [10+x_0]$; $|x_0| \le 0.5$; $|x_1| \le 0.5$. The closed-loop polynomial now becomes

$$P(s) = s^4 + [7 - x_1]s^3 + [45 + 20u_1 - 10x_1 + x_0]s^2 + [194 + 23u_1 + 20u_0 - 5x_1 + 10x_0]s + [96 + 23u_0 + 5x_0].$$

Since $P(s)$ is a polynomial with perturbations entering affine linearity into its coefficients, existing results in the literature can be applied. For example, using the results in [3], it is easy to verify that $P(s)$ is strictly stable for all admissible perturbations.

Analysis Using Theorem 4.1 for Perturbations in U(s), V(s), X(s), and Y(s)

We now consider perturbations in both $U(s)/X(s)$ and $V(s)/Y(s)$. To illustrate computations, we address the following question. For the fixed perturbation bounds on $U(s)$ and $X(s)$ above, what size perturbations can be tolerated in $V(s)$ and $Y(s)$? We illustrate with the perturbations in $V(s)$ and $Y(s)$ confined to a box. Indeed, we consider $V(s) = (20 + v_1)s + (23 + v_0)$ and $Y(s) = s^2 + (10 + y_1)s + (5 + y_0)$ with $|v_i| \le \bar{q}$ and $|y_i| \le \bar{q}$ for $i = 0, 1$. We seek the largest value of $\bar{q}$, call it $\bar{q}_{\max}$, for which robust stability of the closed loop is guaranteed. To begin the analysis, we first note that the satisfaction of Assumption 2.2 is immediate because when $u_0 = u_1 = v_0 = v_1 = x_0 = x_1 = y_0 = y_1 = 0$, $P(s)$ is strictly stable. Then, to find the desired $\bar{q}_{\max}$, we implemented a line search algorithm with respect to $\bar{q}$ on an Apollo computer: at each $\bar{q}$, we performed a frequency sweep while checking Conditions 4.1.1 and 4.1.2. The final result which we obtained was $\bar{q}_{\max} \approx 0.18$. For the reader interested in validating our computations, we show that the system is "barely" unstable for $\bar{q} = 0.19$ since when $w = 5.444$ and $\theta = 3.77427$ the interval (0.55074, 0.55087) is not covered by the $I_i(\theta, w)$.

VI. Concluding Remarks

One of the main technical novelties of this note was the covering condition and its role in robust stability analysis. For the general case when $\mathcal{U}$, $\mathcal{V}$, $\mathcal{X}$, and $\mathcal{Y}$ are *polytopes of polynomials*, current research is aimed at generalization of Conditions 4.1.1 and 4.1.2.

One should note that one of the main strengths of our new approach is the avoidance of the so-called combinatoric explosion associated with edge analysis. That is, one trivial way to obtain results at the polytope level is sketched as follows: letting $U^i(s)$, $V^j(s)$, $X^k(s)$, and $Y^l(s)$ denote the ith, jth, kth, and lth extreme polynomials of $\mathcal{U}$, $\mathcal{V}$, $\mathcal{X}$, and $\mathcal{Y}$, respectively, then robust stability of $P(s) = U(s)V(s) + X(s)Y(s)$ can be studied by considering all polynomials of the following form:

$$[\alpha U^{i_1}(s) + (1-\alpha)U^{i_2}(s)][\beta V^{j_1}(s) + (1-\beta)V^{j_2}(s)] + [\gamma X^{k_1}(s) + (1-\gamma)X^{k_2}(s)][\delta Y^{l_1}(s) + (1-\delta)Y^{l_2}(s)]$$

where $\alpha, \beta, \gamma, \delta \in [0, 1]$. By "pushing" the analysis along this line, one obtains a *double edge theorem*. Clearly, the combinatorics associated with this approach will be even worse than those associated with edge analysis for the "ordinary" polytope of polynomials.

Appendix

Formulas for $R_i^-(\theta, w)$ and $R_i^+(\theta, w)$

As discussed in Section III the formulas used to generate the intervals $I_i(\theta, w)$ for $i = 1, 2, \cdots, 16$ are given below.

$$R_{1i}^-(\theta, w) \doteq \begin{cases} \max\left\{\dfrac{\operatorname{Re} K_1^U(jw)}{\langle K_i^X(jw), e^{-j\theta}\rangle}, 0\right\} \\ \quad \text{if } \langle K_i^X(jw), e^{-j\theta}\rangle < 0; \\ 0 \quad \text{if } \langle K_i^X(jw), e^{-j\theta}\rangle \ge 0; \end{cases}$$

$$R_{1i}^+(\theta, w) \doteq \begin{cases} \max\left\{\dfrac{\operatorname{Re} K_1^U(jw)}{\langle K_i^X(jw), e^{-j\theta}\rangle}, 0\right\} \\ \quad \text{if } \langle K_i^X(jw), e^{-j\theta}\rangle > 0; \\ 0 \quad \text{if } \langle K_i^X(jw), e^{-j\theta}\rangle = 0 \text{ and } \operatorname{Re} K_1^U(jw) \le 0; \\ +\infty \quad \text{otherwise}; \end{cases}$$

$$R_{2i}^-(\theta, w) \doteq \begin{cases} \max\left\{\dfrac{\operatorname{Re} K_2^U(jw)}{\langle K_i^X(jw), e^{-j\theta}\rangle}, 0\right\} \\ \quad \text{if } \langle K_i^X(jw), e^{-j\theta}\rangle > 0; \\ 0 \quad \text{if } \langle K_i^X(jw), e^{-j\theta}\rangle \le 0; \end{cases}$$

$$R_{2i}^+(\theta, w) \doteq \begin{cases} \max\left\{\dfrac{\operatorname{Re} K_2^U(jw)}{\langle K_i^X(jw), e^{-j\theta}\rangle}, 0\right\} \\ \quad \text{if } \langle K_i^X(jw), e^{-j\theta}\rangle < 0; \\ 0 \quad \text{if } \langle K_i^X(jw), e^{-j\theta}\rangle = 0 \text{ and } \operatorname{Re} K_2^U(jw) \ge 0; \\ +\infty \quad \text{otherwise}; \end{cases}$$

$$R_{3i}^-(\theta, w) \doteq \begin{cases} \max\left\{\dfrac{\operatorname{Im} K_3^U(jw)}{\langle K_i^X(jw), e^{j(\pi/2-\theta)}\rangle}, 0\right\} \\ \quad \text{if } \langle K_i^X(jw), e^{j(\pi/2-\theta)}\rangle < 0; \\ 0 \quad \text{if } \langle K_i^X(jw), e^{j(\pi/2-\theta)}\rangle \geq 0; \end{cases}$$

$$R_{3i}^+(\theta, w) \doteq \begin{cases} \max\left\{\dfrac{\operatorname{Im} K_3^U(jw)}{\langle K_i^X(jw), e^{j(\pi/2-\theta)}\rangle}, 0\right\} \\ \quad \text{if } \langle K_i^X(jw), e^{j(\pi/2-\theta)}\rangle > 0; \\ 0 \quad \text{if } \langle K_i^X(jw), e^{j(\pi/2-\theta)}\rangle = 0 \text{ and } \operatorname{Im} K_3^U(jw) \leq 0; \\ +\infty \quad \text{otherwise}; \end{cases}$$

$$R_{4i}^-(\theta, w) \doteq \begin{cases} \max\left\{\dfrac{\operatorname{Im} K_4^U(jw)}{\langle K_i^X(jw), e^{j(\pi/2-\theta)}\rangle}, 0\right\} \\ \quad \text{if } \langle K_i^X(jw), e^{j(\pi/2-\theta)}\rangle > 0; \\ 0 \quad \text{if } \langle K_i^X(jw), e^{j(\pi/2-\theta)}\rangle \leq 0; \end{cases}$$

$$R_{4i}^+(\theta, w) \doteq \begin{cases} \max\left\{\dfrac{\operatorname{Im} K_4^U(jw)}{\langle K_i^X(jw), e^{j(\pi/2-\theta)}\rangle}, 0\right\} \\ \quad \text{if } \langle K_i^X(jw), e^{j(\pi/2-\theta)}\rangle < 0; \\ 0 \quad \text{if } \langle K_i^X(jw), e^{j(\pi/2-\theta)}\rangle = 0 \text{ and } \operatorname{Im} K_4^U(jw) \geq 0; \\ +\infty \quad \text{otherwise}; \end{cases}$$

$$R_{5i}^-(\theta, w) \doteq \begin{cases} \max\left\{\dfrac{\langle K_i^U(jw), e^{j\theta}\rangle}{\operatorname{Re} K_1^X(jw)}, 0\right\} & \text{if } \operatorname{Re} K_1^X(jw) > 0; \\ 0 & \text{if } \operatorname{Re} K_1^X(jw) \leq 0; \end{cases}$$

$$R_{5i}^+(\theta, w) \doteq \begin{cases} \max\left\{\dfrac{\langle K_i^U(jw), e^{j\theta}\rangle}{\operatorname{Re} K_1^X(jw)}, 0\right\} \\ \quad \text{if } \operatorname{Re} K_1^X(jw) < 0; \\ 0 \quad \text{if } \operatorname{Re} K_1^X(jw) = 0 \text{ and } \langle K_i^U(jw), e^{j\theta}\rangle \geq 0; \\ +\infty \quad \text{otherwise}; \end{cases}$$

$$R_{6i}^-(\theta, w) \doteq \begin{cases} \max\left\{\dfrac{\langle K_i^U(jw), e^{j\theta}\rangle}{\operatorname{Re} K_2^X(jw)}, 0\right\} \\ \quad \text{if } \operatorname{Re} K_2^X(jw) < 0; \\ 0 \quad \text{if } \operatorname{Re} K_2^X(jw) \geq 0; \end{cases}$$

$$R_{6i}^+(\theta, w) \doteq \begin{cases} \max\left\{\dfrac{\langle K_i^U(jw), e^{j\theta}\rangle}{\operatorname{Re} K_2^X(jw)}, 0\right\} \\ \quad \text{if } \operatorname{Re} K_2^X(jw) > 0; \\ 0 \quad \text{if } \operatorname{Re} K_2^X(jw) = 0 \text{ and } \langle K_i^U(jw), e^{j\theta}\rangle \leq 0; \\ +\infty \quad \text{otherwise}; \end{cases}$$

$$R_{7i}^-(\theta, w) \doteq \begin{cases} \max\left\{\dfrac{\langle K_i^U(jw), e^{j(\pi/2+\theta)}\rangle}{\operatorname{Im} K_3^X(jw)}, 0\right\} & \text{if } \operatorname{Im} K_3^X(jw) > 0; \\ 0 & \text{if } \operatorname{Im} K_3^X(jw) \leq 0; \end{cases}$$

$$R_{7i}^+(\theta, w) \doteq \begin{cases} \max\left\{\dfrac{\langle K_i^U(jw), e^{j(\pi/2+\theta)}\rangle}{\operatorname{Im} K_3^X(jw)}, 0\right\} \\ \quad \text{if } \operatorname{Im} K_3^X(jw) < 0; \\ 0 \quad \text{if } \operatorname{Im} K_3^X(jw) = 0 \text{ and } \langle K_i^U(jw), e^{j(\pi/2+\theta)}\rangle \geq 0; \\ +\infty \quad \text{otherwise}; \end{cases}$$

$$R_{8i}^-(\theta, w) \doteq \begin{cases} \max\left\{\dfrac{\langle K_i^U(jw), e^{j(\pi/2+\theta)}\rangle}{\operatorname{Im} K_4^X(jw)}, 0\right\} \\ \quad \text{if } \operatorname{Im} K_4^X(jw) \leq 0; \\ 0 \quad \text{if } \operatorname{Im} K_4^X(jw) \geq 0; \end{cases}$$

$$R_{8i}^+(\theta, w) \doteq \begin{cases} \max\left\{\dfrac{\langle K_i^U(jw), e^{j(\pi/2+\theta)}\rangle}{\operatorname{Im} K_4^X(jw)}, 0\right\} \\ \quad \text{if } \operatorname{Im} K_4^X(jw) > 0; \\ 0 \quad \text{if } \operatorname{Im} K_4^X(jw) = 0 \text{ and } \langle K_i^U(jw), e^{j(\pi/2+\theta)}\rangle \leq 0; \\ +\infty \quad \text{otherwise}; \end{cases}$$

for $i = 1, 2, 3, 4$. Using these functions, we define

$$R_k^-(\theta, w) \doteq \max_{1 \leq i \leq 4} R_{ki}^-(\theta, w)$$

$$R_k^+(\theta, w) \doteq \min_{1 \leq i \leq 4} R_{ki}^+(\theta, w).$$

This yields

$$I_k \doteq (R_k^-(\theta, w), R_k^+(\theta, w))$$

for $k = 1, 2, 3, 4, 5, 6, 7, 8$.

To generate $I_9(\theta, w)$ through $I_{16}(\theta, w)$, we repeat the construction above using $Y(s)$ instead of $U(s)$ and $-V(s)$ instead of $X(s)$.

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