Bifurcations in nonlinear systems: computational issues

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Abstract

This paper examines computational issues associated with singular Jacobian and Hopf bifurcations in large nonlinear systems. Singular Jacobian bifurcations can be computed effectively using the Point of Collapse method. Hopf bifurcations can be computed using a Kronecker sum to reduce the problem to the computation of a singular Jacobian. However, this is shown to be computationally intensive. An alternative method for the computation of Hopf bifurcations based on the Point of Collapse method together with a complex expansion of the problem is presented.

Keywords: Computation, bifurcation, sparse matrices, sparsity.

1 Introduction

Nonlinear system bifurcations are often associated with system failure. Two types of bifurcations can lead to failure: the system Jacobian can become singular, or it can acquire a complex eigenvalue pair with a zero real part (a Hopf bifurcation). The boundary between well-behaved cases and unstable cases occurs when the real part of one of the eigenvalues for the system Jacobian matrix becomes zero.

A computational method for the determination of bifurcation points when the instability occurs due to a singular Jacobian has been described by Alvarado and Jung [2] and Ajjarapu [1]. This method is called the Point of Collapse (POC) method, in reference to the original application where the objective was to determine the load level at which the voltage in a power system collapses. This paper examines computational aspects of the POC method and extends its use to complex bifurcations.

Bifurcations associated with complex eigenvalues, however, do not result in a singular Jacobian and therefore the POC method is not applicable directly. A method for reducing a complex bifurcation to the singularity of a matrix based on [8] based on Kronecker sums [6,5] is described. Computational aspects of using this formulation are then examined.

The Kronecker procedure for determining bifurcations is seen to be computationally undesirable. An alternative procedure based on converting the problem to the complex domain is then presented and examined.

Because computational issues are most important in large systems, this paper emphasizes large system examples. Because most large systems result in sparse Jacobians, the example considered is for a sparse system. For a comprehensive review of sparse matrix methods, see [7].

2 The Point of Collapse Method

Consider first a parametrized set of linear differential equations:

$$\dot{x} = A(p)x \tag{2.1}$$

Because this is a linear system, a bifurcation occurs at a value of the parameter p that makes the matrix A singular. One approach to the determination of this value of p is based on the definition of singularity. The existence of a nontrivial vector y that satisfies $Ay = 0$ is required. Since the scaling of y is arbitrary, any norm can be applied to insure that y is nonsingular:

$$A(p)y = 0 \tag{2.2a}$$
$$||y|| = 1 \tag{2.2b}$$

Simultaneous solution of these $n+1$ (now nonlinear) equations for both the vector y and the scalar p yields a value of p that results in a bifurcation. If we consider the 20×20 matrix illustrated in Figure 1 as an example, the topology of the equations that must be solved to solve the problem above is illustrated in Figure 2. The dense right border is associated with the parameter p. This border is not normally full as illustrated. It depends on which terms within A does the parameter p occur. Likewise, if a simpler requirement than a norm is used to insure nontriviality of y, the bottom border becomes sparse. The computationally ideal case is when the nontriviality requirement from equation 2.2b is replaced by $y_i = 1$.

The point of collapse method can be applied to nonlinear systems as well. Consider the nonlinear equation:

$$\dot{x} = f(x,p) \tag{2.3}$$

The point of collapse method is based on the simultaneous solution of the following set of equations:

$$f(x,p) = 0 \tag{2.4a}$$
$$f_x(x,p)y = 0 \tag{2.4b}$$
$$||y|| = 1 \tag{2.4c}$$

These equations are once again based on the definition of singularity. The method implied by these equations is reasonably efficient: the original system equations must be solved simultaneously with the conditions that denote singularity. Assume Figure 1 illustrates the topology of the Jacobian of f. Figure 3 illustrates the topology of the Jacobian for the augmented POC method equations 2.4a–2.4c. This is the densest possible topology. It is quite possible that for specific systems, many of the entries illustrated as nonzeros could in reality be zero. This is problem- specific.

Effective use of the POC method requires the computation of the symbolic Jacobian linearization of a nonlinear system. These symbolic equations must be appended to the original system equations for a simultaneous solution of both sets of equations. The computation of the symbolic Jacobian linearization of a nonlinear set of equations can be done quite effectively using a symbolic equation handling program. SOLVER-Q, a symbolically assisted numeric equation handling environment developed by this author [3,4] is an extremely effective way to deal with symbolic Jacobian computations.

3 Kronecker Sums

When a bifurcation is due to a complex eigenvalue (as in a Hopf bifurcation) the POC method does not work because the system bifurcates without the Jacobian ever becoming singular. This section

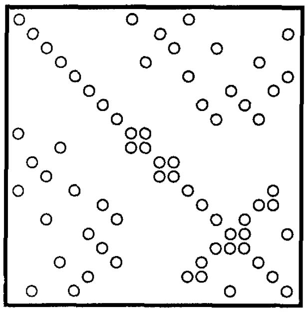

Figure 1: Topology of sample 20 × 20 A matrix.

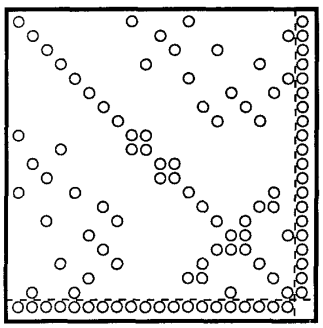

Figure 2: Topology of matrix for POC method, linear case.

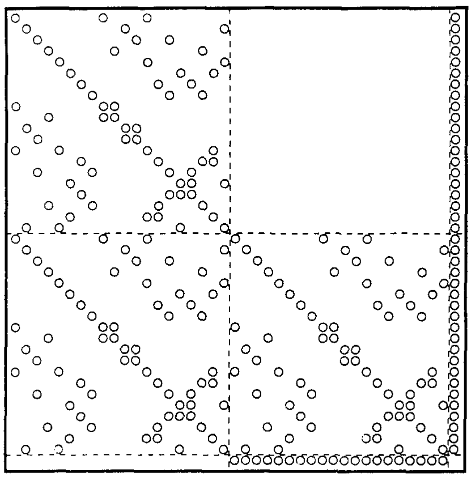

Figure 3: Topology of matrix for POC method, nonlinear case.

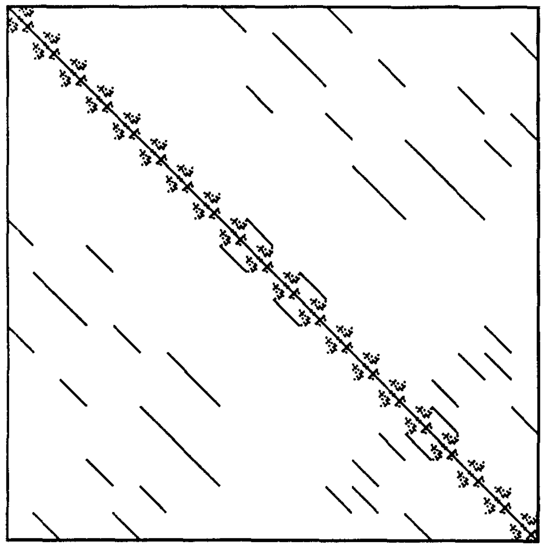

Figure 4: Topology of $A \oplus A$ matrix.

explores one method for the computation of these complex bifurcation points.

Computation of a Hopf bifurcation can be based on Kronecker sums. A Kronecker sum $\oplus$ of a matrix A of dimension $n \times n$ with itself results in a $n^2 \times n^2$ matrix defined as follows [6]:

$$A \oplus A = A \otimes I_n + I_n \otimes A \tag{3.1}$$

Where the symbol $\otimes$ denotes the Kronecker product:

$$A \otimes A = \begin{bmatrix} a_{11}A & a_{12}A & \cdots & a_{1n}A \\ a_{21}A & a_{22}A & \cdots & a_{2n}A \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}A & a_{n2}A & \cdots & a_{nn}A \end{bmatrix} \tag{3.2}$$

The eigenvalues of the Kronecker sum of two matrices A and B is given by the sums of all pairs $\lambda_i(A) + \lambda_j(B)$ for all values of i and j. Therefore, the Kronecker sum of a matrix with itself results in n^2 eigenvalues $\lambda_i(A) + \lambda_j(A)$. A Hopf bifurcation results in a complex eigenvalue pair on the real axis. The Kronecker sum therefore results in an eigenvalue at the origin corresponding to the sum of the two complex ones. In fact, a double eigenvalue at the origin results, one from a ij complex conjugate pair, and one from a ji complex conjugate pair.

The sparsity pattern of the Kronecker matrix for the example from Figure 1 is illustrated in Figure 4. The computational requirements for the solution of equations involving this Jacobian by the POC method require a further expansion of this matrix.

In the Kronecker method, the dimensionality and the density of the matrices increases dramatically. A way to reduce the dimensionality of the Kronecker sum method is to eliminate the redundancy in the Kronecker sum formulation. This can be done using the so-called lower Schläflian matrix, which is of dimension $\frac{1}{2}n(n+1) \times \frac{1}{2}n(n+1)$. The order of this matrix remains n^2.

4 The Complex Formulation

The problem with the Kronecker sum formulation is that it introduces not only a Jacobian singularity for a Hopf bifurcation, but introduces many other additional eigenvalues as well. This section explores alternative formulations for establishing Hopf bifurcation conditions which are natural extensions of the POC methodology. These are based on a complex formulation of the problem.

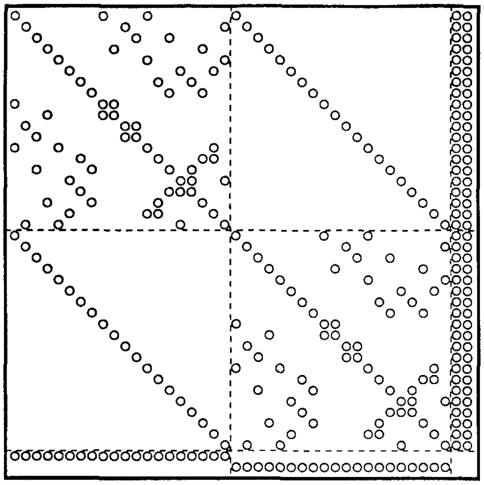

Figure 5: Topology of expanded complex matrix, linear case.

Consider first the linear case. Any eigenvalue of the matrix $A(p)$, even if complex, must satisfy the condition:

$$A(p)y = \lambda y \tag{4.1}$$

Of course, y (the eigenvector associated with the corresponding eigenvalue) is in general complex. Thus, expressing the above relation entirely in terms of real quantities:

$$A(p)(y^r + jy^i) = (\lambda^r + j\lambda^i)(y^r + jy^i) \tag{4.2}$$

A bifurcation occurs when a nontrivial solution to this now nonlinear equation is found for $\lambda^r = 0$. This condition can be expressed as a set of entirely real equations as follows:

$$A(p)y^r = -\lambda^i y^i \tag{4.3a}$$
$$A(p)y^i = \lambda^i y^r \tag{4.3b}$$
$$||y^r|| + ||y^i|| = 1 \tag{4.3c}$$

where $y^r = \Re e(y)$ (the real part of y) and $y^i = \Im m(y)$ (the imaginary part of y).

Figure 5 illustrates the topology pattern of this expanded matrix. Once again, it is not necessary for the norm to use all terms in y or for p to exist in every row in A, which enhances the sparsity of A.

The nonlinear case is based on both a complex expansion of the problem and the POC method formulation. The POC method conditions based on the definition of singularity require the following conditions:

$$f(x,p) = 0 \tag{4.4}$$
$$f_x(x,p)y = \Im(\lambda)y \tag{4.5}$$
$$||\Re(y)|| = 1 \tag{4.6}$$
$$||\Im(y)|| = 1 \tag{4.7}$$

It has been required that $\lambda^r = 0$. Equation 4.5 is complex because both y and λ are complex. The equations can be re-written entirely in terms of real quantities as follows:

$$f(x,p) = 0 \tag{4.8a}$$
$$f_x(x,p)\, y^r = -\lambda^i\, y^r \tag{4.8b}$$

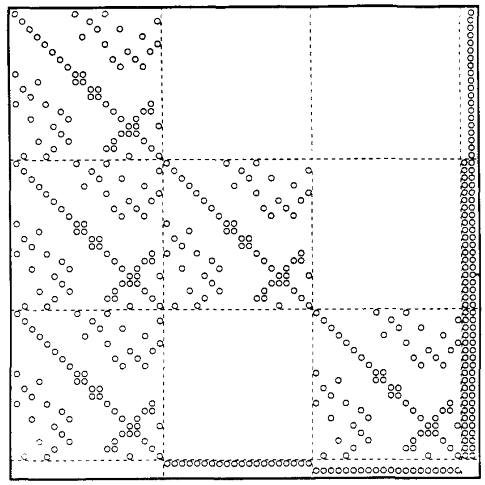

Figure 6: Topology of Jacobian of expanded complex problem, nonlinear case.

Table I: Computational requirements.			
Figure	Dimension	Multiplications	Nonzeros
2	21	264	145
3	41	938	429
4	400	621496	21326
5	42	1678	508
6	62	1920	804

$$f_x(x,p)\, y^i = \lambda^i\, y^i \tag{4.8c}$$
$$||y^r|| = 1 \tag{4.8d}$$
$$||y^i|| = 1 \tag{4.8e}$$

5 Conclusions

The dimension of the Jacobian matrix and its sparsity pattern are important for bifurcation point computations. The use of Kronicker sums, although theoretically pleasing, leads to computational difficulties. The method introduced in this paper, based on an extension of the point of collapse method, results in a potentially more efficient approach to computation. The use of symbolic computational capabilities is of value when using the Point of Collapse method.

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