A Necessary and Sufficient Condition for High-Frequency Robustness of Non-Strictly-Proper Feedback Systems

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Abstract

We consider stability and robustness of feedback systems, where plant and compensator need not be strictly proper. In an earlier paper [1] we described a functional $R_\infty$ which, when negative, guarantees closed-loop instability as a result of parasitic interactions in the feedback loop. In our main result, Theorem 5, we prove that, when $R_\infty > 0$, there exist perturbations of plant and compensator from a narrow class which result in closed-loop stability and convergence. Hence, we may view $R_\infty > 0$ as a necessary and sufficient condition for closed-loop robustness in non-strictly-proper feedback loops.

1 Introduction

Consider the multivariable feedback system in Figure 1, where $P(s)$ and $C(s)$ are matrices of rational functions

$$R = \det(I + CP)$$

and its high-frequency limit

$$R_\infty = \lim_{\sigma \to \infty} R(\sigma) \in [-\infty, \infty].$$

The closed-loop system in Figure 1 is governed by the transfer function matrix

$$H = \begin{bmatrix} P(I + CP)^{-1} & -P(I + CP)^{-1}C \\ (I + CP)^{-1}CP & (I + CP)^{-1}C \end{bmatrix}.$$ 

Henceforth, we adopt the assumption that $H$ is BIBO stable. In particular, this implies that $H$ is proper, so

$$(I + CP)^{-1} = I - C \left( P(I + CP)^{-1} \right)$$

is proper, $R$ is not strictly proper, and $R_\infty \neq 0$.

A natural approach to studying stability and robustness of Figure 1 is to examine strictly proper perturbations of $P$ and $C$. Then conventional feedback theory can be utilized. For technical reasons, we need additional assumptions. We say that $P_k \to P$ weakly if there exists $\sigma < \infty$ such that

W1) $P_k$ has no pole in $[\sigma, \infty)$ for large $k$;

W2) $P_k \to P$ pointwise on $[\sigma, \infty)$.

Suppose we construct weakly convergent sequences $P_k \to P$ and $C_k \to C$. Letting $R_k = \det(I+C_k P_k)$, it is obvious from the definition of weak convergence that $R_k \to R$ weakly. The zeros of $R$ are poles of $H$, and the zeros of $R_k$ are poles of the perturbed closed-loop system $H_k$. In [1] we prove the following result.

Theorem 1 If $P_k$ and $C_k$ are strictly proper, $P_k \to P$ and $C_k \to C$ weakly, and $R_\infty < 0$, then there exist $\sigma_k \in \mathbb{R}$ such that $\sigma_k \uparrow \infty$ and $R_k(\sigma_k) = 0$ for every $k$.

Theorem 1 says that, under very mild assumptions, $R_\infty < 0$ guarantees that $H_k$ has a high-frequency pole $\sigma_k$ on the positive real axis, guaranteeing extreme instability of the closed-loop system. Our objective in this paper is to show that, when $R_\infty > 0$, we have the opposite situation — viz. that the closed-loop system is robust to certain reasonable perturbations of $P$ and $C$. 

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2 Preliminaries

We begin by recalling some basic facts about rational matrices and their state-space realizations. The characteristic polynomial $\Delta_p$ of a rational matrix $P$ is the least common denominator of all minors of $P$. If $P$ is strictly proper, its McMillan degree is $\nu(P) = \deg \Delta_p$. $\Delta_p$ is also the characteristic polynomial of any state-space realization of minimal dimension (i.e., any controllable and observable realization). Appropriate extensions of realization theory to the case of non-strictly-proper $P$ are developed in [3] and summarized in [4], Theorem 1.2. If $P$ is non-strictly-proper, we may perform entrywise polynomial division to obtain $P = P_s + P_f$, where $P_s$ is strictly proper and $P_f$ is polynomial. Let $R$ be the operator on the space of rational matrices defined by

$$ R(P)(s) = -\frac{1}{s} P \left( \frac{1}{s} \right). $$

Then $R(P_f)$ is strictly proper and we may define the degree of $P$ according to

$$ \mu(P) = \nu(P_s) + \nu(R(P_f)). $$

Our analysis hinges on state-space realizations of $P$ and $C$. Suppose $P$ has realization

$$ A \dot{x} + B u_1 = \begin{bmatrix} E & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & -B & H \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \quad (1) $$

$$ y_1 = C \begin{bmatrix} x \\ z \end{bmatrix} \quad (2) $$

with minimal dimension. (See [8] for basic information on singular systems.) Then $P(s) = C(sE - A)^{-1} B$ and, from [3], $(E, A, B, C)$ is a controllable and observable 4-tuple with $\mu(P)$ states. The characteristic polynomial of (1) is

$$ \Delta_p(s) = \det(sE - A). $$

It can be shown that

$$ \deg \Delta_p \leq \text{rank } E $$

with equality iff $P$ is proper. Applying the Weierstrass decomposition to (1) (see [5], Ch.12), we obtain

$$ M_p E N_p = \begin{bmatrix} I_{n_s} & 0 \\ 0 & A_f \end{bmatrix}, \quad M_p A N_p = \begin{bmatrix} A_s & 0 \\ 0 & I_{n_f} \end{bmatrix}, $$

$$ M_p B = \begin{bmatrix} B_s \\ B_f \end{bmatrix}, \quad C N_p = \begin{bmatrix} C_s & C_f \end{bmatrix}, $$

where $M_p$ and $N_p$ are nonsingular and $A_f$ is nilpotent. Letting

$$ \begin{bmatrix} x_s \\ x_f \end{bmatrix} = N_p^{-1} x $$

leads to the decoupled state-space system

$$ \begin{bmatrix} I_{n_s} & 0 \\ 0 & A_f \end{bmatrix} \begin{bmatrix} \dot{x}_s \\ \dot{x}_f \end{bmatrix} = \begin{bmatrix} A_s & 0 \\ 0 & I_{n_f} \end{bmatrix} \begin{bmatrix} x_s \\ x_f \end{bmatrix} + \begin{bmatrix} B_s \\ B_f \end{bmatrix} u_1 $$

$$ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} C_s & 0 \\ 0 & C_f \end{bmatrix} \begin{bmatrix} x_s \\ x_f \end{bmatrix}, $$

$$ y_1 = C \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} x_s \\ x_f \end{bmatrix}. $$

Then (1) has transfer function matrix

$$ \begin{bmatrix} C_s & C_f \end{bmatrix} \begin{bmatrix} x_s \\ x_f \end{bmatrix}, $$

and characteristic polynomial

$$ \Delta_p(s) = \alpha \det(sI - A_s) \det(sI - A_f) $$

for some constant $\alpha \neq 0$. Note that $P$ is proper iff $A_f = 0$.

Similar statements can be made about $C$, yielding a realization

$$ J \dot{z} = F z + G u_2 $$

$$ y_2 = H z, $$

a Weierstrass decomposition

$$ M_c J N_c = \begin{bmatrix} I_{n_c} & 0 \\ 0 & F_f \end{bmatrix}, \quad M_c F N_c = \begin{bmatrix} F_s & 0 \\ 0 & I_{n_f} \end{bmatrix} $$

$$ M_c G = \begin{bmatrix} G_s & G_f \end{bmatrix}, \quad H N_c = \begin{bmatrix} H_s & H_f \end{bmatrix}, $$

and characteristic polynomial

$$ \Delta_c(s) = \beta \det(sI - F_s) \det(sI - F_f) $$

Then $H$ has minimal realization

$$ \begin{bmatrix} E & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} A & -B & H \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} $$

$$ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & H \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}. $$

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The closed-loop characteristic polynomial is

$$
\Delta_{cl}(s) = \det \begin{bmatrix}
    sI - A_s & B_sH_s & 0 & B_sH_f \\
    G_sC_s & sI - F_s & -G_sC_f & 0 \\
    0 & B_fH_s & B_fH_f & 0 \\
    -G_fC_s & 0 & -G_fC_f & sF_f - I
\end{bmatrix}
$$

BIBO stability of $H$ implies properness of $H$; so, as in (2), $\deg \Delta_{cl} = n_{ps} + n_{ca} + p$, where

$$
\rho = \text{rank} A_f + \text{rank} F_f.
$$

From [6], p.159, we know that

$$
R = \frac{\Delta_{cl}}{\Delta_{pa} \Delta_{ca}}.
$$

Let

$$
\gamma_0 s^p + \cdots + \gamma_0 = \det \begin{bmatrix}
    I & -sA_f & -B_fH_f \\
    G_fC_f & I & -sF_f
\end{bmatrix}.
$$

Recalling

$$
\det (sA_f - I) = (-1)^{n_f}, \quad \det (sF_f - I) = (-1)^{n_f},
$$

we obtain

$$
R_{\infty} = \begin{cases}
    \gamma_0, & \text{P and C proper}, \\
    \gamma_0, & \text{P or C improper}.
\end{cases}
$$

Note that $P$ and $C$ proper implies

$$
R_{\infty} = \det \begin{bmatrix}
    I & -B_fH_f \\
    G_fC_f & I
\end{bmatrix} = \det (I + B_fH_f G_fC_f).
$$

3 Sufficiency of $R_{\infty} > 0$

Merely showing that $R_{\infty} > 0$ guarantees closed-loop robustness to certain weak perturbations would not be an acceptable result, since the class of weak perturbations is so large. To obtain a better result, we limit our analysis to the narrowest perturbation class normally encountered in singular perturbation problems. As an initial step, we consider rational functions

$$
f_k(s) = \frac{b_{2k}s^2 + \cdots + b_{0k}}{a_{rk}s^r + \cdots + a_{0k}}
$$

and say that $f_k \to f$ parametrically if

$$
q \geq m, \quad r \geq n
$$

and say that $f_k \to f$ parametrically if

$$
\begin{align*}
    a_{ik} &\to a_i; & i = 0, \ldots, n - 1 \\
    a_{nk} &\to 1 \\
    a_{ik} &\to 0; & i = n + 1, \ldots, r \\
    b_{ik} &\to b_i; & i = 0, \ldots, m \\
    b_{nk} &\to 0; & i = m + 1, \ldots, q.
\end{align*}
$$

For matrices, we say $P_k \to P$ parametrically if each entry converges parametrically. We say $P_k \to P$ strongly, if

$$
\begin{align*}
    &\text{S1} \quad P_k \to P \text{ parametrically,} \\
    &\text{S2} \quad \mu(P_k) = \mu(P) \text{ for large } k, \\
    &\text{S3} \quad \text{there exists } \epsilon > 0 \text{ and } K < \infty \text{ such that, when } k > K, \text{ no finite pole } \lambda_{ik} \text{ of } P_k \text{ satisfies } |\lambda_{ik}| > \frac{1}{\epsilon} \text{ and } |\arg \lambda_{ik}| < \frac{\pi}{2} + \epsilon.
\end{align*}
$$

Condition S3 is equivalent to saying that the divergent poles of $P_k$ lie in a fixed left half-plane sector. S1) and S3) together imply weak convergence. Furthermore, it is shown in [4] that S1) guarantees that $P_k$ and $C_k$ have realizations of the form (1) with convergent matrices. Let $L^{-1}$ denote the inverse Laplace transform operator. Then, from S3) and [7], Theorem 1, $L^{-1} \{P_k\} \to L^{-1} \{P\}$ and $L^{-1} \{C_k\} \to L^{-1} \{C\}$ as distributions. (See [2] for a discussion of distributional convergence.) In addition, we can show that the inverse transforms converge uniformly on compact subintervals of $(0, \infty)$. Hence, strong convergence embodies all the properties that are normally encountered in classical singular perturbation problems. We will eventually prove that, for $R_{\infty} > 0$, the closed-loop system is robust to certain strong, strictly proper plant and compensator perturbations.

Next we study a class of perturbations of $P$ and $C$ obtained by choosing $A_{f_k} \to A_f$ and $F_{f_k} \to F_f$ and substituting $A_{f_k}$ and $F_{f_k}$ into (7) and (5) in place of $A_f$ and $F_f$. Recall that the index ind $A$ of a square matrix $A$ is the smallest integer $p \geq 1$ such that rank $A^p = $ rank $A^{p+1}$. It is easy to show that ind $A = 1$ is equivalent to having rank $A$ nonzero eigenvalues in $A$, counting multiplicities.

**Lemma 2** Let $A_{f_k} \to A_f$ and $F_{f_k} \to F_f$, where rank $A_{f_k} = $ rank $A_f$, rank $F_{f_k} = $ rank $F_f$, ind $A_{f_k} = $ ind $F_{f_k} = 1$, and every nonzero eigenvalue $\lambda_{ik}$ of $A_{f_k}$ and $F_{f_k}$ satisfies $\text{Re} \lambda_{ik} < 0$ for large $k$. Then $P_k$, $C_k$, and $H_k$ are proper and $R_{\infty} \to R_{\infty}$.

**Proof.** If $P$ and $C$ are proper, then $A_f = A_{f_k} = 0$ and $F_f = F_{f_k} = 0$, so $P_k = P$, $C_k = C$, $H_k = H$, and

$$
R_{\infty} = R_{\infty}.
$$

Suppose $P$ and $C$ are not both proper. Since $A_{f_k}$ and $F_{f_k}$ have unit index, the corresponding $P_k$ and $C_k$ are proper. Despite the fact that $A_{f_k}$ and $F_{f_k}$ may not be nilpotent, we may substitute them for $A_f$ and $F_f$ in (3), (6), and (7), yielding $\Delta_{p_k}, \Delta_{c_k},$ and $\Delta_{c_k}$. Applying (2) to (7), we obtain properness of $H_k$. Applying (8) to the perturbed system, we obtain

$$
R_{\infty} = \lim_{s \to \infty} \Gamma_k(s),
$$

where

$$
\Gamma_k(s) = \frac{\det \begin{bmatrix}
    sA_{f_k} - I & B_{f_k}H_f \\
    -G_{f_k}C_f & sF_{f_k} - I
\end{bmatrix}}{\det (sA_{f_k} - I) \det (sF_{f_k} - I)} = \gamma_0 s^p + \cdots + \gamma_0
$$

$$
\prod_{i=1}^k (1 - \lambda_{ik}s),
$$

(11)
with \( \gamma_{ik} \rightarrow \gamma_i \). Thus

\[
R_{\infty} = \frac{\gamma_{jk}}{\prod_{i=1}^p (\lambda_{ik})}.
\] (12)

Since the denominator of (12) is positive, real, and converging to 0, \( R_{\infty} \rightarrow \gamma_p \cdot \infty = R_{\infty} \).

Strongly convergent sequences \( P_k \) and \( C_k \) satisfying the conditions of Lemma 2 are easily constructed. For example, let \( P \) have minimal realization (??), and suppose \( A_{f} \) has Jordan form

\[
T^{-1}A_{f}T = \begin{bmatrix}
J_1 & & \\
& \ddots & \\
& & J_l
\end{bmatrix}, \quad J_i = \begin{bmatrix}
0 & 1 \\
& \ddots & 1 \\
& & 0
\end{bmatrix}
\]

Let

\[
A_{f} = T \begin{bmatrix}
J_{1k} & & \\
& \ddots & \\
& & J_{lk}
\end{bmatrix} T^{-1},
\]

\[
J_{ik} = \begin{bmatrix}
-\frac{1}{k} & 1 \\
& \ddots & 1 \\
& & -\frac{1}{k}
\end{bmatrix}
\]

Lemma 3 Suppose \( A_{f} \) and \( F_{f} \) are constructed as in (13) and (14). Then \( P_k, C_k, \text{ and } H_k \) are proper, \( P_k \rightarrow P, \text{ and } C_k \rightarrow C, \) and \( H_k \rightarrow H \) strongly and \( R_{\infty} \rightarrow R_{\infty} \).

Proof. The conditions of Lemma 2 obviously hold, guaranteeing properness of \( P_k, C_k, \text{ and } H_k \) and convergence of \( R_{\infty} \). Also, (S1) and (S2) are obvious for \( P_k, C_k, \) and \( H_k \). The divergent poles of \( P_k \) and \( C_k \) are just \( \lambda_{ik} = -k \), so (S3) holds. From (2) and (7),

\[
n_{ps} + n_{cs} \geq \deg \Delta_{ck} \geq \deg \Delta_{ci} = n_{ps} + n_{cs}
\]

so \( \deg \Delta_{ck} \) is constant. Hence \( H_k \) has no divergent poles and (S3) holds vacuously.

Before we state our main theorem, we need one more preliminary result.

Lemma 4 A square matrix \( M \) is the product of two stable matrices iff \( \det M > 0 \) and \( M \neq -\alpha I \) for any \( \alpha > 0 \).

Proof. (Necessary) Let \( M = \Sigma \Pi, \) where \( \Sigma \) and \( \Pi \) are stable with eigenvalues \( \{\sigma_i\} \) and \( \{\pi_i\} \), respectively. Then

\[
\det M = \left( \prod \sigma_i \right) \left( \prod \pi_i \right) > 0.
\]

If \( M = -\alpha I \) with \( \alpha > 0 \), then \( \Sigma^{-1} = -\frac{1}{\alpha} \Pi \) is unstable. Stability of \( \Sigma \) yields a contradiction.

(Sufficient) A proof of the converse is too long to present here in detail. The general idea is to first construct a nonsingular matrix \( T \) such that every leading principal minor of \( T^{-1}MT \) is positive. Second, find a lower triangular triangular \( \Sigma \) and an upper triangular \( \Pi \) such that \( T^{-1}MT = \Sigma \Pi \). These constructions can be performed using standard matrix manipulations and an inductive argument.

Theorem 5 If \( R_{\infty} > 0 \), then there exist strictly proper sequences \( P_k \) and \( C_k \) such that \( P_k \rightarrow P, \text{ and } C_k \rightarrow C, \) and \( H_k \rightarrow H \) strongly and \( H_k \) is proper for large \( k \).

Proof. Our construction proceeds in four stages. First, we assume \( P \) is strictly proper and \( C \) is proper. This means \( n_{ps} = 0 \) and \( F_{f} = 0 \). Let \( P_k = P \) and \( C_k \) be determined by setting \( F_{jk} = \frac{1}{\Sigma} \Sigma_{1} \), where \( \Sigma_{1} \) is any stable matrix. Then \( C_k \) and \( H_k \) are obviously strictly proper and satisfy (S1) and (S2). Since the divergent poles of \( C_k \) are just the eigenvalues of \( \Sigma^\infty \), \( \Sigma_{1} \) follows for \( C_k \). From [9], Corollary 2.1, \( H_k \) satisfies (S3).

Now assume that \( P \) and \( C \) are proper and that \( I + B_f H_{f} G_{f} C_{f} \neq -\alpha I \) for any \( \alpha > 0 \). Then, by Lemma 4 and (10), there exist stable matrices \( \Sigma \) and \( \Pi \) such that \( I + B_f H_{f} G_{f} C_{f} = \Sigma \Pi, \) so \( \Sigma^{-1} \Pi \) is stable. Let \( A_{f} = \frac{1}{\Sigma} \Sigma_1 \) and fix \( F_{f} = 0 \). Then \( P_k \) is strictly proper and \( P_k \rightarrow P \) strongly. Letting \( C_k = C \), [9], Corollary 2.1, again implies \( H_k \rightarrow H \) strongly and \( H_k \) proper. Based on the construction in the preceding paragraph, for each \( k \) we can find a sequences \( P_{kj} \rightarrow P \) and \( C_{kj} \) such that \( C_{kj} \) is strictly proper and \( C_{kj} \rightarrow C_k \) and \( H_{kj} \rightarrow H_k \) strongly as \( j \rightarrow \infty \) for every \( k \). Hence, there exists a sequence of integers \( j_k \in \infty \) such that \( P_{kj_k}, C_{kj_k}, \text{ and } H_{kj_k} \) are strongly convergent.

Next, assume \( P \) and \( C \) are proper, but \( I + B_f H_{f} G_{f} C_{f} \neq -\alpha I \) for some \( \alpha > 0 \). Then there exists a sequence \( B_{f,k} \rightarrow B_f \) such that \( I + B_{f,k} H_{f} G_{f} C_{f} \neq -\beta_k I \) for any \( \beta_k > 0 \). Fix \( A_{f} = 0 \) and \( F_{f} = 0 \), but use \( B_{f,k} \) in place of \( B_{f} \) in (??). Then \( P_k \) is proper, \( \Sigma = C, \) and \( P_k \rightarrow P \) strongly, since its poles are constant. As in the proof of Lemma 3, \( \deg \Delta_{ck} \) is constant; hence, \( H_k \rightarrow H \) strongly with \( H_k \) proper. As in the preceding paragraph, we may construct \( P_{kj}, C_{kj}, \text{ and } j_k \) such that \( P_{kj_k}, C_{kj_k}, \text{ and } H_{kj_k} \) are strongly convergent.

Finally, suppose \( P \) and \( C \) are not both proper. Then, from Lemma 3, the construction (13) and (14) yields strongly convergent proper sequences \( P_k, C_k, \) and \( H_k \) with \( R_{\infty} \rightarrow R_{\infty} \). From the preceding paragraph, we may construct \( P_{kj}, C_{kj}, \text{ and } j_k \) such that \( P_{kj_k}, C_{kj_k}, \text{ and } H_{kj_k} \) are strongly convergent.

We note that condition (S3) along with properness of \( H_k \) imply BIBO stability of \( H_k \) for large \( k \). Hence, the construction in the proof of Theorem 5 yields strong perturbations of \( P \) and \( C \) under which the closed-loop system is robustly stable. This establishes sufficiency of \( R_{\infty} > 0 \).
References


