Nonlinear tearing mode study using the ''almost ideal magnetohydrodynamics (MHD)'' constraint

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Tearing modes of finite amplitude are studied numerically using the ''almost ideal magnetohydrodynamics (MHD)'' constraint, which is a modification of the ideal MHD constraints that allows reconnection of the magnetic field lines. For the one-dimensional initial equilibria studied here, the stability criterion is found to be the same as that of the linear tearing mode theory, namely $\Delta' < 0$. The nonlinear saturation level of the mode can also be determined; it is found to be smaller than that estimated from $\Delta'(W_{sat}) = 0$ [R. B. White *et al.*, Phys. Fluids **20**, 800 (1977)].
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I. INTRODUCTION

The tearing mode[1] is an important resistive magnetohydrodynamics (MHD) mode. It perturbs the initial equilibrium magnetic flux surfaces through magnetic field line reconnection to form new flux surfaces with magnetic islands. In the study of the tearing mode,[1,2] usually the initial equilibria are one-dimensional with two ignorable coordinates, and the perturbed equilibria are two-dimensional with one ignorable coordinate. Therefore, magnetic flux surfaces exist for both the initial and the perturbed equilibria (Fig. 1). The tearing mode can be linearly unstable.[1] For the linearly unstable case, White, Monticello, Rosenbluth, and Waddell have made an analytical theory for the nonlinear saturation of the unstable mode.[3] More recently, a neoclassical tearing mode theory[4] shows that the mode can be nonlinearly driven by the bootstrap current even when it is linearly stable to the classical tearing mode. There exists a threshold amplitude above which the mode can be nonlinearly excited. The neoclassical tearing mode has been experimentally observed.[5] However, the origin of the threshold is not definitely resolved.[6,7]

As an intrinsically nonlinear approach, the use of the ''almost ideal MHD'' constraint[8] is suited to study the nonlinear properties of the tearing mode. In this paper, as a validation of the method, we study two characteristics of the tearing mode using the ''almost ideal MHD'' constraint: (1) the linear stability condition for the initial one-dimensional equilibrium; and (2) the final saturation level for the unstable case. In this work, we only consider the simplest case where no gradient of pressure or current density exists at the mode resonant surface.

The tearing mode cannot exist under the ideal MHD constraint, which does not permit changes in flux surface topology.[9] On the other hand, it is well known that the ideal MHD constraint is violated only in a narrow region near the mode resonant surface, where the magnetic field is perpendicular to the wave vector of the mode.[1] The ''almost ideal MHD'' constraint is a relaxation of the ideal MHD constraint to allow such local changes of the flux surface topology.[9]

In principle, one can find many equilibria with islands which are associated with the initial one-dimensional equilibrium through the ''almost ideal MHD'' constraint. In general, these island equilibria have a singular current at the X-point of the islands [Fig. 1(b)]. Depending on the sign of this singular current, the equilibrium evolves, on the tearing mode time scale, in the direction of either increasing or decreasing the island width. We are interested in an equilibrium where the singular current is zero and therefore is a final stationary state. If such a final equilibrium is found to include an island (it is believed that there is at most one such equilibrium), then the initial equilibrium is presumed to be tearing-mode unstable and the island width of the final state is the saturation width. If no possible final state with an island is found, then the initial equilibrium is tearing-mode stable. Because we do not use any small amplitude expansion, this approach is intrinsically nonlinear.

In the rest of the paper, we use an example to illustrate the application of the ''almost ideal MHD'' constraint to the problem of the tearing mode stability and saturation. We will show that the stability criterion is the same as the linear ''Δ' theory''[1] and give the saturation width for an unstable initial equilibrium. As mentioned earlier, this is only done for simple initial equilibria.

II. AN EXAMPLE

We consider a simple case where there is no pressure gradient, $\nabla p = 0$. Then, the force-balance equation for the equilibrium is just $\mathbf{J}\times\mathbf{B} = 0$. We use Cartesian coordinates where $\hat{\mathbf{z}}$ is an ignorable coordinate $(\partial/\partial z = 0)$ for both the

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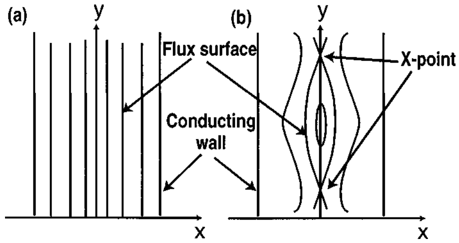

FIG. 1. Magnetic flux surfaces for two equilibria: (a) initial one-dimensional equilibrium. (b) Final two-dimensional equilibrium with island.

initial and final equilibrium. The initial equilibrium is also independent of y. The magnetic field $\mathbf{B}$ is expressed through functions $B_z(x,y)$ and $\psi(x,y)$ as

$$\mathbf{B}=B_z\hat{\mathbf{z}}+\nabla\psi\times\hat{\mathbf{z}}.$$

Then, the force-balance equation becomes

$$B_z\nabla B_z+(\nabla^2\psi)\nabla\psi+\hat{\mathbf{z}}(\nabla\psi\times\nabla B_z\cdot\hat{\mathbf{z}})=0.$$

The $\hat{\mathbf{z}}$-component of the above equation gives $B_z=B_z(\psi)$. The rest of the equation is

$$\nabla^2\psi+B_z\frac{dB_z}{d\psi}=0. \tag{1}$$

Equation (1) determines both the initial and final equilibrium, once $B_z(\psi)$ and the boundary conditions are given. For the boundary conditions, we assume that there are two perfectly-conducting walls at $x=\pm 1$ where $\psi(\pm 1,y)=0$. The boundary condition in the y-direction is set to be periodic with a period of L, $\psi(x,y)=\psi(x,y+L)$.

The one-dimensional initial equilibrium can be described by $\psi=\psi_i(x)$. For simplicity, we further assume $\psi_i(x)=\psi_i(-x)$ and that ψ_i decreases monotonically from the center ($x=0$) toward the two conducting walls ($x=\pm 1$). The resonant surface is at $x=0$ where ψ reaches its maximum value. By assuming left-right symmetry, we only have to study the $x\geqslant 0$ region. The approach can be generalized to the asymmetrical case.[8]

One family of such initial equilibria can be described by $B_zB_z'=a^2\psi+b$, which results in $\psi=-(b/a^2)(1-\cos ax/\cos a)$, with $0<a<\pi$ and $b/\cos a>0$. The value of Δ', which is the key linear stability parameter,[1] for these initial equilibria can readily be found analytically, yielding $\Delta'=-2\kappa\cot\kappa$ with $\kappa^2=a^2-k^2$ and $k=2\pi/L$. A positive Δ' means instability.

The ideal MHD constraint for this $\nabla p=0$ case can be represented by requiring that the B_z flux enclosed by a flux surface $\psi=\psi_1$,

$$\phi(\psi_1)=\int B_zS[\psi(x,y)-\psi_1]dxdy,$$

$$S[u]=\begin{cases}1, & u>0,\\ 0, & u<0,\end{cases}$$

be a conserved function. In a tokamak, this is equivalent to the conservation of the safety factor. For simplicity, we consider the limit where $B_z\to\infty$, $B_z'\to 0$, and B_zB_z' remains finite. Then, the conservation of $\phi(\psi)$ is equivalent to the incompressibility of the plasma. That is, the area enclosed by a flux surface remains constant; i.e.,

$$A(\psi_1)=\int S[\psi(x,y)-\psi_1]dxdy$$

is a conserved function.

To relax this constraint to the ''almost ideal MHD'' constraint, we only require a finite number of ''moments'' of $A(\psi)$ to be conserved. That is, the set of integrals,

$$g_n=\int G_n(\psi)dA,\quad n=1,2,\ldots,N,$$

where G_n is a set of base functions, are conserved. The base functions G_n should be sufficiently diverse in ψ-space. The number of base functions is finite to allow magnetic field line reconnection. When the number of base functions approaches infinity, the ''almost ideal MHD'' constraint approaches the ideal MHD constraint. We choose the following set of the ''tent'' base functions:

$$\frac{dG_n}{d\psi}=\begin{cases}(N-1)[\psi/\psi_{max}-(n-2)/(N-1)], & \text{for } (n-2)/(N-1)\leqslant\psi/\psi_{max}\leqslant(n-1)/(N-1),\\ 1-(N-1)[\psi/\psi_{max}-(n-1)/(N-1)], & \text{for} (n-1)/(N-1)\leqslant\psi/\psi_{max}\leqslant n/(N-1),\\ 0, & \text{otherwise,}\end{cases}$$

where ψ_{max} is the maximum value of ψ. So formally, the ''almost ideal MHD'' constraint can be written as

$$g_{nf}=g_{ni},\quad n=1,2,\ldots,N \tag{2}$$

where g_{nf} and g_{ni} are the integrals of the nth base function over the final and initial state, respectively. The set g_{ni} can be calculated from the given initial equilibrium and is the only information needed from it. Our goal is to find a final state $\psi(x,y)$ that satisfies both Eq. (1) and Eq. (2), by choosing the right form of B_zB_z'.

III. NUMERICAL ALGORITHM

Finding the exact solution of Eqs. (1) and (2) is difficult. Therefore, a numerical algorithm is used to find an approximate solution. The function B_zB_z' is parameterized by a set of parameters p_m, $m=1,2,\ldots,M$. The aim is to adjust p_m's to minimize the quantity $Q=\Sigma_{n=1}^{N}(g_{nf}-g_{ni})^2$. The quantity Q is a function of the p_m's and $dQ\approx\Sigma_{m=1}^{M}(\partial Q/\partial p_m)dp_m$. If the p_m's change an amount $dp_m=-s(\partial Q/\partial p_m)$ with step s being a sufficiently small positive number, then Q decreases an amount of $dQ\approx-s\Sigma_m(\partial Q/\partial p_m)^2$. Given a set of p_m's, one can calculate the Q and $\partial Q/\partial p_m$'s by first solving Eq. (1) and then finding the g_{nf}'s from the solution of Eq. (1). This procedure of decreasing Q can be repeated until Q is minimized. The final ψ thus obtained is the final equilibrium of the initial state and it can then be examined to see whether it has an island structure.

Obviously, the initial island-free equilibrium itself is always a possible final state. A successful algorithm must find a final state other than the initial one if the initial equilibrium is unstable to the tearing mode. A proof is given in the appendix showing that the Jacobi algorithm[10] (or its variants like the Gauss-Seidel and Successive Over-Relaxation algorithms) used to solve Eq. (1) converges to the initial state only if the Δ' of the initial state is negative. So if the initial equilibrium is tearing-mode unstable ($\Delta'>0$), the algorithm moves the solution away from it towards a different equilibrium.

Numerically, Eq. (1) is solved on a 25 by 49 grid with a grid size of $\Delta x=\Delta y=1/24$. The length of the grid in the y-direction is half of the period L plus $2\Delta y$ (so $L=23/6$) and the boundary conditions in the y-direction are $\partial\psi(x,0)/\partial y=\partial\psi(x,L/2)/\partial y=0$. The number of base functions N is taken to be 14. The form of B_zB_z' is taken to be

$$B_zB_z'=p_1+p_2\left(1-\frac{\psi}{\psi_{max}}\right)^{1/2}+\sum_{i=3}^{M}p_i\left(1-\frac{\psi}{\psi_{max}}\right)^{i-2},$$

with $M=7$. The final result is found not to depend sensitively on the grid size, the number of base functions (N), or the number of the parameters (M) in the expression of B_zB_z'.

The relative amount of reconnected flux, $\delta\psi\equiv1-\psi_x/\psi_{max}$, is a measure of the size of the island. Here, ψ_x is the ψ value at the X-point. The following relation may be used to relate $\delta\psi$ and the full island width W_{sat} when the island width is small,

$$\delta\psi=\frac{1}{2}\left(\frac{W_{sat}}{2}\right)^2|\psi_i''/\psi_i|_{x=0}, \tag{3}$$

where ψ_i is the ψ of the initial equilibrium. Plotted in Fig. 2 is $\delta\psi$ calculated using the algorithm described here for the family of initial equilibria (different Δ') described in Section II. We can see that the island only exists ($\delta\psi>0$) when $\Delta'>0$. Thus, the "almost ideal MHD" approach gives the same linear stability criterion as conventional tearing mode theory.[1] As a comparison, the saturation island width can also be estimated using the assumption that the mode saturates when the value of the nonlinear version of the stability parameter Δ' vanishes.[3,11] For the family of the initial equilibrium used in our example, the saturation width is found analytically to be

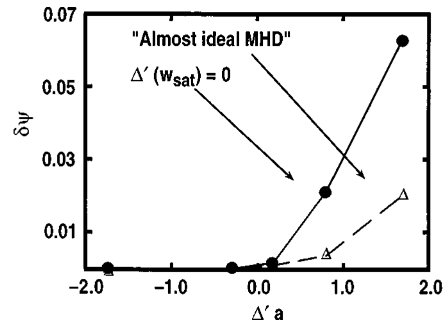

FIG. 2. The reconnected flux $\delta\psi$ for different initial equilibria (different Δ') calculated by using the "almost ideal MHD" constraint and by using the requirement $\Delta'(W_{sat})=0$. In both cases, an island exists ($\delta\psi>0$) only when $\Delta'>0$.

$$W_{sat}=-\frac{2}{\kappa}\arctan(\cot\kappa). \tag{4}$$

The $\delta\psi$ estimated from Eqs. (3) and (4) is also plotted in Fig. 2. The saturation level from the "almost ideal MHD" approach is found to be somewhat smaller than that estimated from the combination of Eqs. (3) and (4).

IV. DISCUSSION

We have demonstrated in this paper that the "almost ideal MHD" constraint can be used to determine linear stability of an equilibrium of slab geometry against the tearing mode. We have also demonstrated that the constraint can be used to determine the saturation amplitude for the linearly unstable case. The saturation amplitude found differs somewhat from that predicted by the theory of White *et al.*[3] We do not know the origin of this discrepancy.

The initial equilibria studied here are symmetric and have no pressure gradient. The "almost ideal MHD" constraint approach can be extended to include a pressure or a current density gradient at the resonant surface of the initial equilibrium.[8] It is possible that under these more complicated situations linearly stable but nonlinearly unstable cases, similar to the neoclassical tearing mode, can be found.[12]

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APPENDIX: RELATION BETWEEN THE NUMERICAL AND PLASMA INSTABILITY

The Jacobi algorithm is

$$\frac{d\psi}{dN}=\frac{d^2}{4}[\nabla^2\psi+f(\psi)],$$

where N is the iteration number, d is the grid size, and $f(\psi)\equiv B_zB_z'$. Near the desired solution ψ_s we expand ψ as $\psi=\psi_s+\phi$ with $|\phi|\ll|\psi_s|$. Then the change of ϕ after each iteration is

$$\frac{d\phi}{dN}=\frac{d^2}{4}[\nabla^2\phi+f'(\psi_s)\phi].$$

The convergence of the scheme is equivalent to the requirement that all eigenvalues λ_k in the eigenvalue problem (with appropriate boundary conditions),

$$\nabla^2\phi_k+f'(\psi_s)\phi_k+\lambda_k\phi_k=0, \tag{A1}$$

are positive.

The linearized equation which allows a discontinuity in the first derivative of the tearing mode amplitude ψ_1 at the resonant surface $(x=0)$ is

$$\nabla^2\psi_1+f'(\psi_s)\psi_1+c\,\delta(x)=0, \tag{A2}$$

where $\delta(x)$ is the δ-function and c is a constant. The stability parameter Δ' is, by definition,

$$\Delta'\equiv\frac{1}{\psi_1(0)}\int_{0^-}^{0^+}\psi_1''dx=-\frac{c}{\psi_1(0)}. \tag{A3}$$

We can expand $\delta(x)$ and $\psi_1(x)$ in the eigenfunctions ϕ_k:

$$\delta(x)=\sum a_k\phi_k,$$

$$\psi_1(x)=\sum b_k\phi_k. \tag{A4}$$

Comparing Eqs. (A1) and (A2) we obtain

$$\frac{ca_k}{b_k}=\lambda_k. \tag{A5}$$

The coefficients a_k can be determined from the properties of $\delta(x)$ and the orthogonality of ϕ_k, which gives

$$a_k=\frac{\phi_k(0)}{d_k^2}, \tag{A6}$$

with $d_k^2=\int\phi_k^2dx$. Putting Eqs. (A4), (A5), and (A6) into (A3), we obtain

$$\Delta'=-\left(\sum\frac{1}{\lambda_k}\cdot\frac{\phi_k^2(0)}{d_k^2}\right)^{-1}.$$

Therefore, for the Jacobi algorithm to converge to the initial equilibrium (all $\lambda_k>0$), Δ' must be negative.

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