Pretty Peculiar Properties of the Fibonacci Sequence under a Modulous

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Background

The **Fibonacci sequence**, which was first studied by Leonardo Fibonacci, is the sequence recursively defined with initial conditions $F_0 = 0$ and $F_1 = 1$ such that

$$F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2.$$

Later Fibonacci's work was popularized by Édouard Lucas, who had a similar sequence named after him. The first to consider the Fibonacci sequence under a modulus was Wall in 1960, to which we owe many of the properties used in our research.

Definition: The Fibonacci sequence modulo m is defined as the set

$$\{\mathcal{F}_{m,n}\}_{n=0}^{\infty} = \{F_n \pmod m\}_{n=0}^{\infty}$$

Definition: The number of terms, $\pi(m)$, until $\{\mathcal{F}_{m,n}\}_{n=0}^{\infty}$ begins to repeat itself is called the **Pisano period** of $\{\mathcal{F}_{m,n}\}_{n=0}^{\infty}$.

Pisano periods for various m-values.

m	3	4	5	6	10	11	12	15	20	50	100	150	200	500	504
$\pi(m)$	8	6	20	24	60	10	24	40	60	300	300	600	300	1500	48

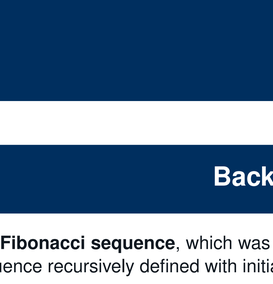
Definition: The sequences $\{\mathcal{F}_{m,rj}\}_{r=0}^{\infty}$ in which all $j \in U(\pi(m))$ yield the parent sequence in either forward or reverse are called m**-complete**.

For any given $\{\mathcal{F}_{m,n}\}_{n=0}^{\pi(m)-1}$ there can only be 1, 2, or 4 zeros, and the number of zeros for a given m-value is denoted as $\beta(m)$.

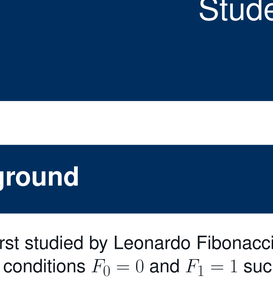
Motivation

We explored properties of the Fibonacci sequence modulo m and its subsequences and attempted to categorize the properties that arise for classes of various m values. To aid us in this categorization, we used generalizations from aBa Mbirika, Dan Guyer, and Miko Scott's investigation of the Fibonacci sequence modulo 10. Additionally, from Renault's paper on the Fibonacci sequence we learned of the formalization of the number of zeros, $\beta(m)$.

In order to better grasp the Fibonacci sequence under a modulus we created circles, such as

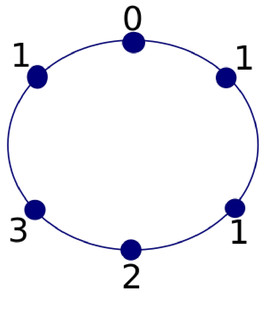


$\{\mathcal{F}_{5,n}\}_{n=0}^{19}$



$\{\mathcal{F}_{6,n}\}_{n=0}^{23}$,

to be able to more easily see patterns. As shown above, for $\beta(5) = 4$, the zeros are all equally spaced on the circle. Additionally, for $\beta(6) = 2$, there are zeros on the top and bottom of the circle. It was shown by Renault that the zeros can only possibly occur at $\mathcal{F}_{m,\frac{k\pi(m)}{4}}$ for $k \in 0, 1, 2, 3$.



$\{\mathcal{F}_{4,n}\}_{n=0}^{5}$

Above, for $\beta(4) = 1$, the zero is located at the top of the circle, at the beginning of the sequence. Also This is the case for any $m \equiv 2 \pmod 4$. These three circles display all possible arrangements of zeros with respect to the pisano period.

Results

As we conducted our research, we noticed many peculiar properties of the Fibonacci sequence modulo m. Some of these properties were able to be precisely determined, such as the following:

Theorem: If $\beta(m) = 4$ for a given Fibonacci subsequence modulo m, then

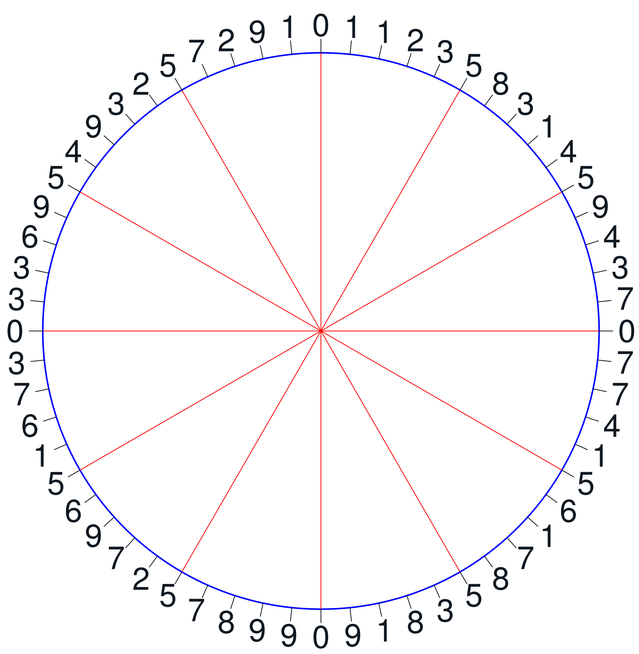
$$1 + \mathcal{F}_{m,\frac{\pi(m)}{2}+1} \equiv 0 \pmod m.$$

That is, if $\beta(m) = 4$, then $\mathcal{F}_{m,\frac{\pi(m)}{2}+1} \equiv m - 1 \pmod m$.

Theorem: For a given $\{\mathcal{F}_{m,rj}\}_{r=0}^{\infty}$, then for $j = \frac{\pi}{2}$ we find that

$$\mathcal{F}_{m,r\frac{\pi(m)}{2}} = \mathcal{F}_{m,\frac{\pi(m)}{2}}, \text{ for all r} \in U(\pi(m)).$$

This theorem is of interest when $\beta(m) = 1$, since $\mathcal{F}_{m,\frac{\pi(m)}{2}} \not\equiv 0 \pmod m$ when $\beta(m) = 1$.



The Pisano period $\{\mathcal{F}_{10,n}\}_{n=0}^{59}$ with values equally spaced

Above is a circle representing the Fibonacci sequence when $m = 10$. This sequence gave us direction to look at certain aspects of the Fibonacci sequence under various different m values.

One of those thing with which we were interested was whether there were other sequences, apart from $\{\mathcal{F}_{10,n}\}_{n=0}^{\infty}$, which had the property of being m-complete. Of all investigated moduli, there were only three m-values which yielded an m-complete sequence, which were $m \in \{5, 6, 10\}$.

In [1] it was noted that the sum of antipodal points for $\{\mathcal{F}_{10,n}\}_{n=0}^{\infty}$. This was what led us to develop the following Theorem,

Theorem: If $\beta(m) = 4$, then for any $r \in \mathbb{Z}$ we find that

$$\mathcal{F}_{m,r} + \mathcal{F}_{m,r+\frac{\pi(m)}{2}} \equiv 0 \pmod m.$$

That is, for any point on a corresponding circle for $\{\mathcal{F}_{m,n}\}_{n=0}^{\infty}$ the addition of that point and the point farthest from it on the circle will result in a multiple of m.

We found that for all observed values of m that if $r \in U(\pi(m))$ then $\mathcal{F}_{m,r} \in U(m)$. We also developed the following **almost** trivial fact,

Theorem: If m is prime, then for all $r \in U(\pi(m))$ it follows that $\mathcal{F}_{m,r} \in U(m)$.

Future Investigation

As we conducted our research, we discovered new peculiar properties every week. Further avenues of research may include

- Determining whether there are more than 3 values for m such that $\{\mathcal{F}_{m,n}\}_{n=0}^{\infty}$ is m-complete.
- Discovering what determines the value at the bottom of the circle, $\mathcal{F}_{m,\frac{\pi(m)}{2}}$, when $\beta(m) = 1$ and why the surrounding values add to m.

Table where $\beta(m) = 1$

m	$\pi(m)$	$\mathcal{F}_{m,\frac{\pi(m)}{2}}$	right value	left value
4	6	2	1	3
11	10	5	3	8
19	18	15	2	17
22	30	16	3	19
29	14	13	8	21
31	30	21	26	5

- Determining why, seemingly, the same one or two numbers appear at every multiple of $\frac{\pi(m)}{6}$ for all $j \in U(\pi(m))$ when $\pi(m)$ is divisible by 6 and $\beta(m) = 1$.
- Prove or disprove that for all values of m, if $r \in U(\pi(m))$ then $\mathcal{F}_{m,r} \in U(m)$.
- If $\beta(m) \neq 4$, are there still interesting properties with the sums of antipodal points?
- Finding a closed form expression for the Pisano period.

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References

[1] D. Guyer, a. Mbirika, and M. Scott: "Tantalizing Properties of the Fibonacci Sequence modulo 10." Submitted.

[2] M.S. Renault: "The Fibonacci sequence under various moduli", Master's Thesis, Wake Forest University (1996).

[3] S. Vajda: *Fibonacci and Lucas Numbers, and the Golden Section*, Chichester, England: Ellis Horwood Limited (1989), 190 pp.

[4] D.D. Wall: "Fibonacci series modulo m". *Amer. Math. Monthly*. **67** No. 6 (1960), 525–532.