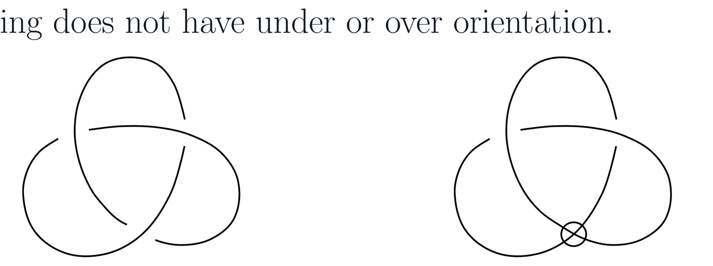
Tait Graphs and Properties of Virtual Knots

Izabel Steinmetz
Faculty Advisor: Dr. Carolyn Otto
University of Wisconsin - Eau Claire

Introduction

The primary objective of our research project is to discover relationships between graph theory and knot theory. We are particularly interested in virtual knots, a knot that has three different types of crossings, and how they relate to their Tait graph. A Tait graph is a graph that is associated to a knot. A way to color a Tait graph for virtual knots is established where classical and virtual crossings can be easily identified. We describe how these graphs are invariant under the Reidemeister moves, both normal and virtual. We also establish a way to produce a Tait graph for normal and virtual doubling operators for a Pure Double, Whitehead Double, or Bing Double of a knot given the original Tait graph. Formulas for the number of edges and vertices of a Tait graph for a doubled knot using the number of edges in the original Tait graph are given.

Knot - A knot is defined to be a simple, closed curve in 3-space.
Virtual knot - A virtual knot is a knot that has both classical and virtual crossings. A virtual crossing does not have under or over orientation.



Trefoil knot Virtual Trefoil knot

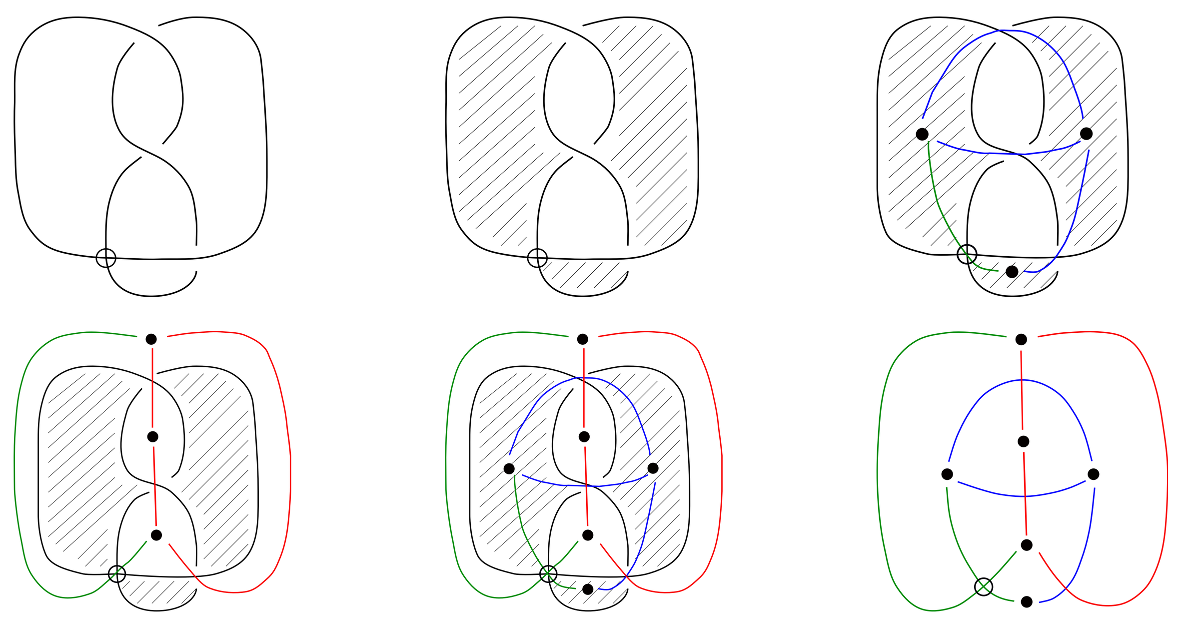
Tait Graphs

Tait graph - A Tait graph is a graph that is associated to a knot. A Tait graph is made by using checkerboard coloring as seen below.

Every knot has two Tait graphs, black and white. We developed a way to create a Tait graph for virtual knots.

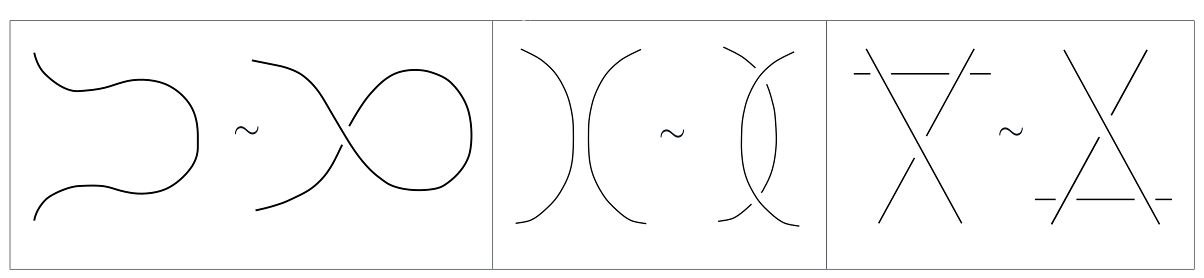
For a black Tait graph, use blue to color the edges where all normal crossings occur and green to color the edges where all virtual crossings occur.

For a white Tait graph, use red to color the edges where all normal crossings occur and green to color the edges where all virtual crossings occur.

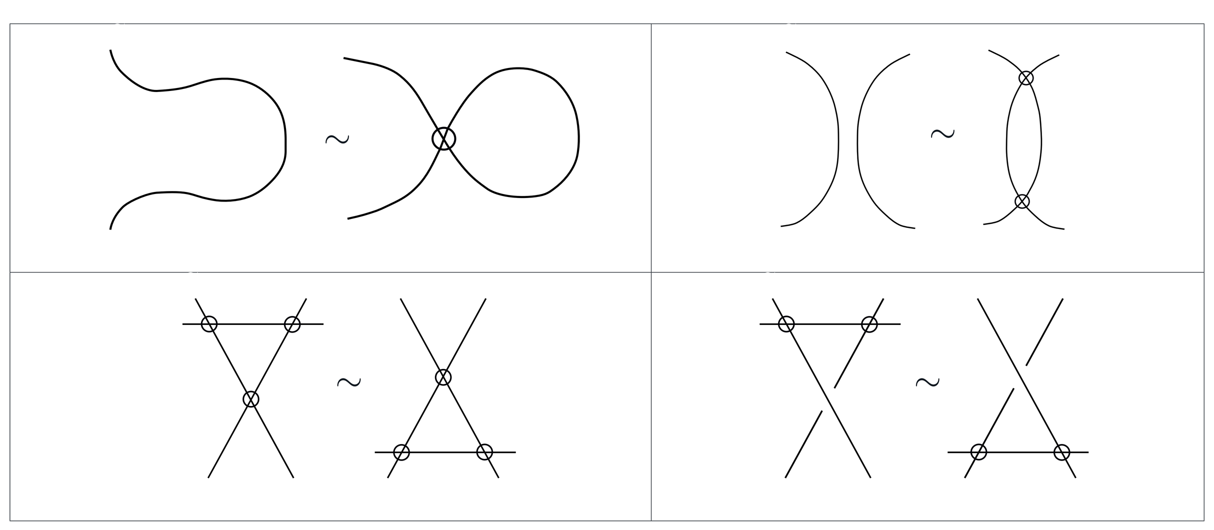


Reidemeister Moves

Reidemeister moves - Reidemeister moves are equivalence moves for knots. The moves are diagrammatic changes that can be made if the knot was made out of string, and they do not affect the underlying diagram. The diagrammatic changes happen in a certain area of the knot, while the rest of the knot remains unchanged.



Reidemeister moves I-III

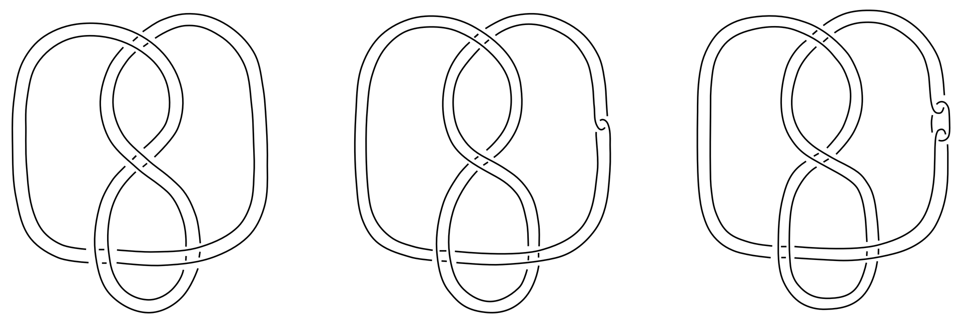


Virtual Reidemeister moves I-IV

Doubling Operators

Doubling operators - A doubling operator is an operation performed on a knot in which all segments are doubled.

There are three doubling operators we investigate. A Pure Double knot is a knot with a doubling operation performed on it. A Whitehead Double knot is a knot with a doubling operation performed on it; however, there is a clasp within the doubled knot. A Bing Double knot is a knot with a doubling operation performed on it, but there is a link within the doubled knot.



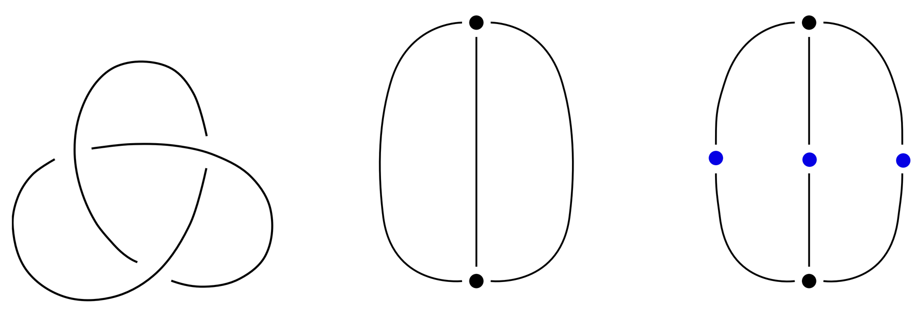
Pure Double Whitehead Double Bing Double

Tait Graphs for Doubling Operators

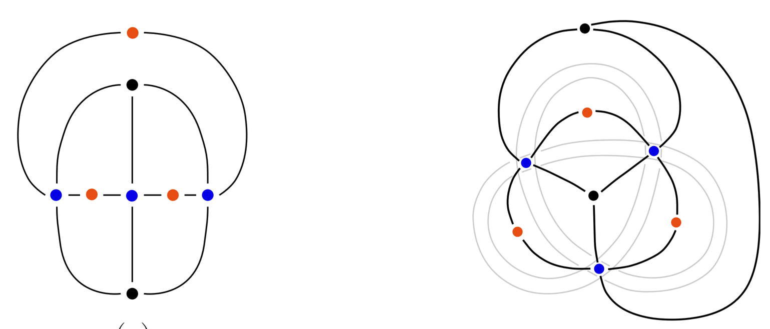
We established ways to create a white Tait graph for each doubling operator, both for normal and virtual knots. The procedure will be illustrated with the trefoil knot, as seen below.

Pure Double knot

1. Let G be a white Tait graph with edges, e_i, and vertices, v_j, where $i \in \{1,...,m_1\}$ and $j \in \{1,...,m_2\}$ and $m_1, m_2 \in \mathbb{Z}$.
2. For every edge in G, add another vertex, w_k, on that edge where $k \in \{1,...,n\}$ and $n \in \mathbb{Z}$. Those newly added vertices represent the region created by having one doubling strand over another in the doubling knot.
3. For every face in G, add a vertex, u_l, where $l \in \{1,...,F\}$ where F is the number of faces. Connect every u_l to every w_k that is within the boundary of the face, meaning no edges should be crossed.



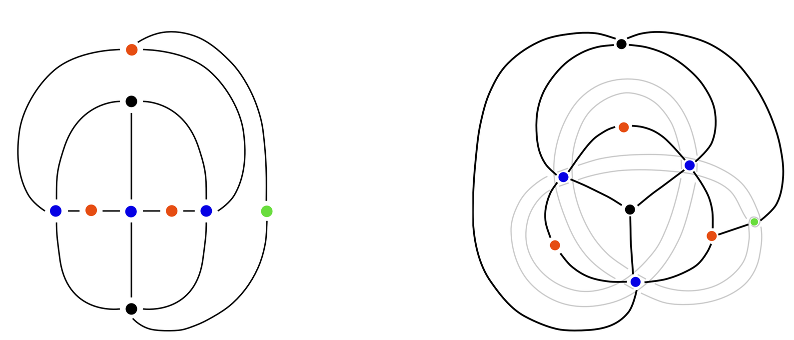
(a) (b)



(c) (d)

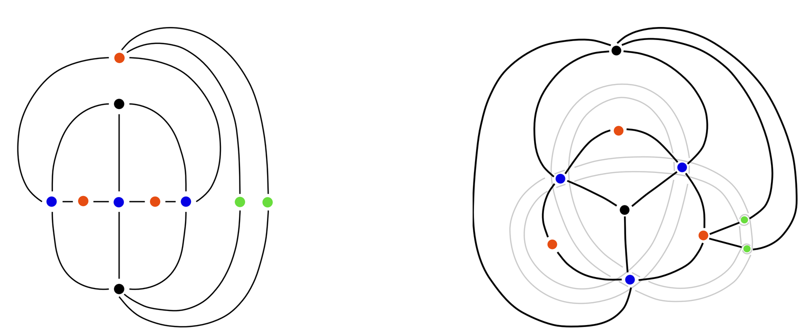
Whitehead Double knot

Steps 1-3 are the same as for creating a white Tait graph for a Whitehead Double knot, reference images a-d above.
4. Add vertices y_c, one for every region created by the clasp in a Whitehead Double knot. Connect y_c to vertices u_l and v_j that are within the boundary of the face.



Bing Double knot

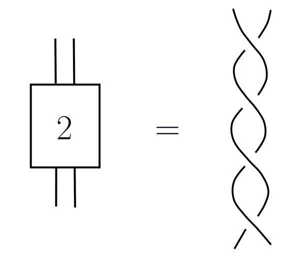
Steps 1-3 are the same as for creating a white Tait graph for a Bing Double knot, reference images a-d above.
4. Add two more vertices, x_1 and x_2. These represent the holes in the link of the Bing Double knot. Connect x_1 and x_2 to vertices u_l and v_j that are within the boundary of the face.



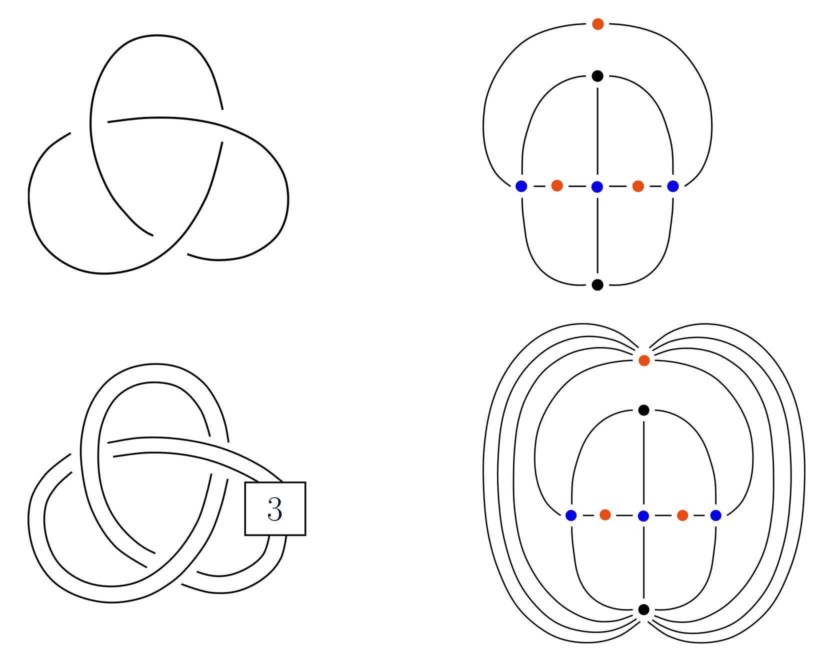
Tait Graphs for Modified Doubling Operators

0-Pure Double knot

A 0-Pure Double knot is a knot with a doubling operation performed on it that has a writhe of 0. We wish to establish a way to create a white Tait graph for a 0-Pure Double knot in four steps. It is important to look at the 0-framings because these are preserved under Reidemeister moves. For ease of visualization, we will simplify the drawing by establishing a box within the knot and placing the number of full twists inside, notice to the right.



1. Let G be a white Tait graph with edges, e_i, and vertices, v_j, where $i \in \{1,...,m_1\}$ and $j \in \{1,...,m_2\}$ and $m_1, m_2 \in \mathbb{Z}$.
2. For every edge in G, add another vertex, w_k, on that edge where $k \in \{1,...,n\}$ and $n \in \mathbb{Z}$. Those newly added vertices represent the region created by having one doubling strand over another in the doubling knot.
3. For every face in G, add a vertex, u_l, where $l \in \{1,...,F\}$ where F is the number of faces. Connect every u_l to every w_k that is within the boundary of the face, meaning no edges should be crossed.
4. Add $2|-w|$ edges, which represent the edges that cross over the $-w$ full twists, where w is the writhe of the doubled knot. Those edges will be connected to a v_j and u_l within the boundary of the face.



Notice that the $-w$ full twists are added to a 0-Pure Double knot. No vertices are added the white Tait graph when you compare it to the Pure Double knot. Only edges are added, in fact, $2|-w|$ edges are added. The 0-Pure Double Trefoil knot has +3 full twists. Thus, the white Tait graph of the 0-Pure Double Trefoil knot has 6 edges added when you compare it to the Pure Double Trefoil knot.

Results

Let e_i be the amount of edges in the original White Tait graph, G, of a knot, K.

Edges	Vertices
Formulas for Pure Doubles and Virtual Pure Doubles of K	
$4e_i$	$2e_i + 2$
Formulas for Whitehead Doubles and Virtual Whitehead Doubles of K	
$4e_i + 2$	$2e_i + 3$
Formulas for Bing Doubles and Virtual Bing Doubles of K	
$4e_i + 4$	$2e_i + 4$
Formulas for 0-Pure Doubles of K	
$4e_i + 2\|-w\|$	$2e_i + 2$
Formulas for 0-Whitehead Doubles of K	
$4e_i + 2 + 2\|-w\|$	$2e_i + 3$
Formulas for 0-Bing Doubles of K	
$4e_i + 4 + 2\|-w\|$	$2e_i + 4$

Acknowledgements and References

- Office of Research and Sponsored Programs, UW-Eau Claire
- Department of Mathematics, UW-Eau Claire
- Graphics computed and rendered using Inkscape
- Dye, Heather A. *An Invitation to Knot Theory: Virtual and Classical*. Boca Raton, CRC Press, 2016. Print.