Efficient Erasure Reconstruction from Noisy Measurements

Tyler Gonzales and Austin Holmes
Advisor: Dr. Sam Scholze
University of Wisconsin-Eau Claire Department of Mathematics

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Motivation and Goal

The goal of this project is to efficiently reconstruct signals when some of the frame measurements are erased or subject to additive noise.

Frames

A finite collection $\{f_j\}_{j=1}^N$ of vectors in $\mathbb{C}^n$ is called a frame for $\mathbb{C}^n$ if there exists positive constants A and B such that for every $f \in \mathbb{C}^n$, $A||f||^2 \leq \sum_{j=1}^N |\langle f, f_j\rangle|^2 \leq B||f||^2$ holds.

- A Parseval frame is one where we can take $A = B = 1$.
- In finite dimensions, frames are the same thing as spanning sets.

Frame Operators

Common Operators associated with frames include the Analysis Operator $F^* : \mathbb{C}^n \to \mathbb{C}^N$ defined as

$$F^*f = \sum_{j=1}^N \langle f, f_j\rangle e_j.$$

The Synthesis Operator $F : \mathbb{C}^N \to \mathbb{C}^n$ is given by

$$F\left[\sum_{j=1}^N c_j e_j\right] = \sum_{j=1}^N c_j f_j.$$

The composition of the Analysis Operator and Synthesis Operator gives us the Frame Operator $S : \mathbb{C}^n \to \mathbb{C}^n$ defined by

$$Sf = FF^*f = \sum_{j=1}^N \langle f, f_j\rangle f_j.$$

Reconstructions

If $\{f_j\}_{j=1}^N$ is a frame for $\mathbb{C}^n$, S^{-1} exists, and for all $f \in \mathbb{C}^n$,

$$f = \sum_{j=1}^N \langle f, S^{-1}f_j\rangle f_j = \sum_{j=1}^N \langle f, f_j\rangle S^{-1}f_j.$$

The canonical dual frame to $\{f_j\}_{j=1}^N$ is given by $\{S^{-1}f_j\}_{j=1}^N$. Any frame $\{g_j\}_{j=1}^N$ satisfying

$$f = \sum_{j=1}^N \langle f, g_j\rangle f_j = \sum_{j=1}^N \langle f, f_j\rangle g_j$$

for all $f \in \mathbb{C}^n$ is called a dual frame to $\{f_j\}_{j=1}^N$. For Parseval frames, $S = I$, and the above becomes

$$f = \sum_{j=1}^N \langle f, f_j\rangle f_j = \sum_{j=1}^N \langle f, f_j\rangle f_j.$$

Each of these expansions is similar to an orthonormal basis expansion, with the added bonus of redundancy.

Alice and Bob

How does Alice send the signal $f \in \mathbb{C}^n$ to Bob given a dual frame pair $(\{f_j\}_{j=1}^N, \{g_j\}_{j=1}^N)$?

- Alice computes the measurements $(\langle f, g_j\rangle)_{j=1}^N$
- Alice transmits the measurements to Bob over some channel
- Bob receives the coefficients and synthesizes with the frame $\{f_j\}_{j=1}^N$

In the end, Bob receives the signal $f = \sum_{j=1}^N \langle f, g_j\rangle f_j$.

Reconstruction from Erasures

Assume that the coefficients indexed by an erasure set, Λ, are erased in the channel. Can Bob still recover the original signal?

- If $\{g_j\}_{j\in\Lambda^c}$ is no longer a frame for $\mathbb{C}^n$, no.
- If $\{g_j\}_{j\in\Lambda^c}$ is still a frame for $\mathbb{C}^n$, yes.
- As a general rule, as long as $|\Lambda| < N - n$, a reconstruction is possible.
- Reduced direct inversion is an efficient method to solve this problem.

We now look at some important operators in this analysis. The Partial Reconstruction operator is defined as

$$R_\Lambda f = \sum_{j\in\Lambda^c} \langle f, g_j\rangle f_j.$$

The Partial reconstruction of $f \in \mathbb{C}^n$ is thus

$$f_R = R_\Lambda f = \sum_{j\in\Lambda^c} \langle f, g_j\rangle f_j.$$

Notice that f_R is the signal Bob receives after accounting for the erasures. Notice to recover f from f_R, invert R_Λ:

$$f = R_\Lambda^{-1} f_R$$

An efficient inversion formula is:

$$f = [I + F_\Lambda(I - G_\Lambda^* F_\Lambda)^{-1} G_\Lambda^*] F_{\Lambda^C}(\langle f, g_j\rangle)_{j\in\Lambda^C}$$

where F_Λ is the synthesis operator for $\{f_j\}_{j\in\Lambda}$ and G_Λ^* is the analysis operator for $\{g_j\}_{j\in\Lambda}$. Notice that R_Λ is an $n \times n$ matrix, whereas $(I - G_\Lambda^* F_\Lambda)$ is an $|\Lambda| \times |\Lambda|$ matrix. Thus, inverting $(I - G_\Lambda^* F_\Lambda)$ is more efficient.

Restricted Isometry Property

Definition: Let Φ be an $m \times N$ matrix, and $s < N$. The Restricted Isometry Constant, δ_s, of order s of Φ is the smallest number such that

$$(1 - \delta_s)\|x\|^2 \leq \|\Phi x\|^2 \leq (1 + \delta_s)\|x\|^2$$

holds for all s-sparse vectors. A vector is called s-sparse if it has at most s nonzero entries.

Noise Term and Error Bounds

Let $\Delta = [I + F_\Lambda(I - G_\Lambda^* F_\Lambda)^{-1} G_\Lambda^*] F_{\Lambda^c}$. Notice that Δ is a linear operator, and if the measurements are subject to an additive noise term, ε, then the reconstructed signal is

$$\begin{aligned}\Delta((\langle f, g_j\rangle)_{j\in\Lambda^c} + \varepsilon) &= \Delta((\langle f, g_j\rangle)_{j\in\Lambda^c}) + \Delta(\varepsilon)\\ &= f + \Delta\varepsilon.\end{aligned}$$

Thus, the error in the reconstruction is simply $\Delta\varepsilon$.

Theorem (Error Bounding)

Let $\{h_j\}_{j=1}^N$ be a Parseval frame for $\mathbb{C}^n$. Set $f_j = \sqrt{\frac{N}{n}}h_j$, and $g_j = \sqrt{\frac{n}{N}}h_j$. Assume that $|\Lambda| < s$, and ε is s-sparse. Let δ_s denote the Restricted Isometry Constant for F (the synthesis operator for F) of order s. Then,

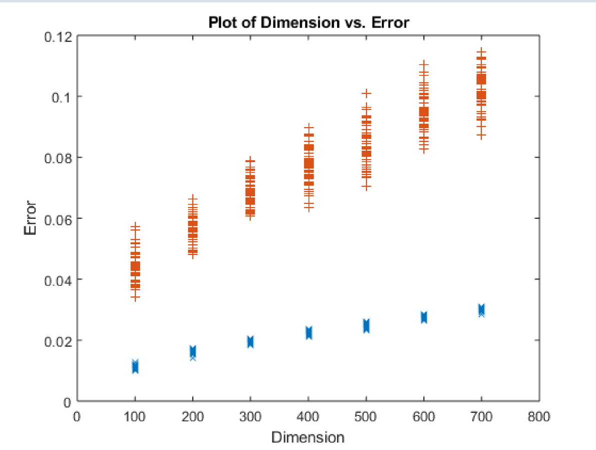
$$\|\Delta\varepsilon\| \leq \frac{N\sqrt{1+\delta_s}}{N - n(1+\delta_s)}\|\varepsilon\| \xrightarrow{N\to\infty} \sqrt{1+\delta_s}\|\varepsilon\|.$$

Computer Experiments and Results

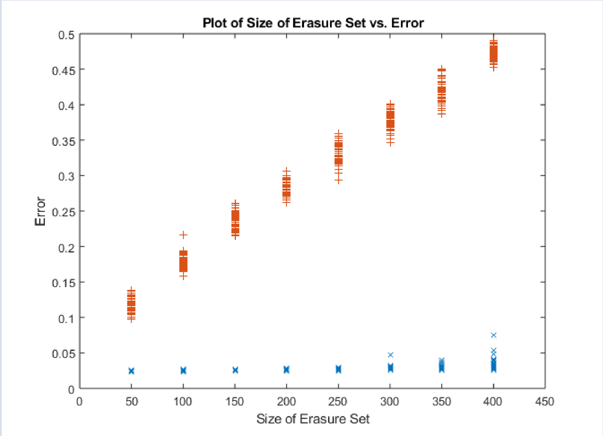
Goal: To better understand the relationship between the signal error, the dimension of the space, the size of the erasure set, and the length of the frame.

- The larger the size of the frame, the smaller the error.
- The larger the erasure set, the larger the error.
- For a fixed frame length, the larger the dimension, the larger the error.
- Performing the reconstruction is almost always better than not performing a reconstruction.

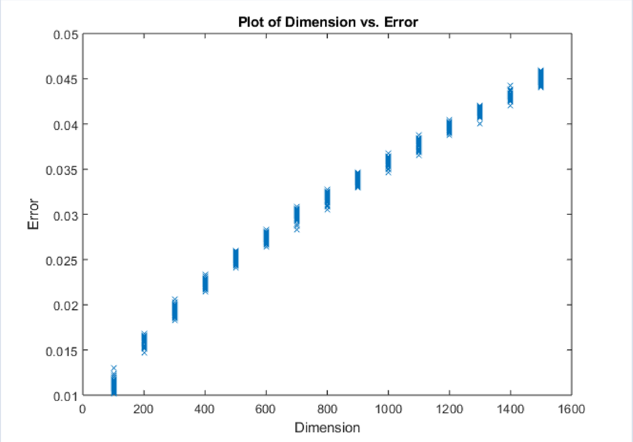
Graph Comparisons

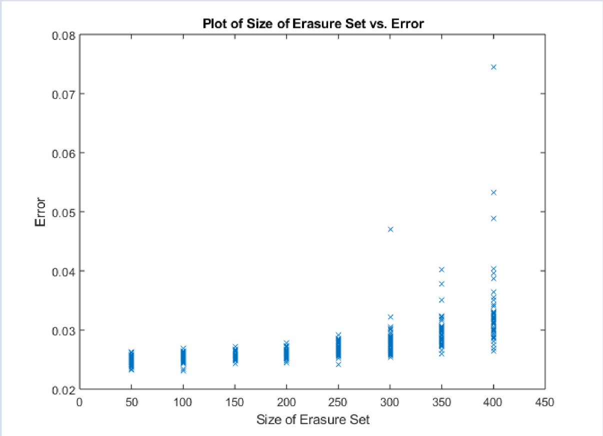


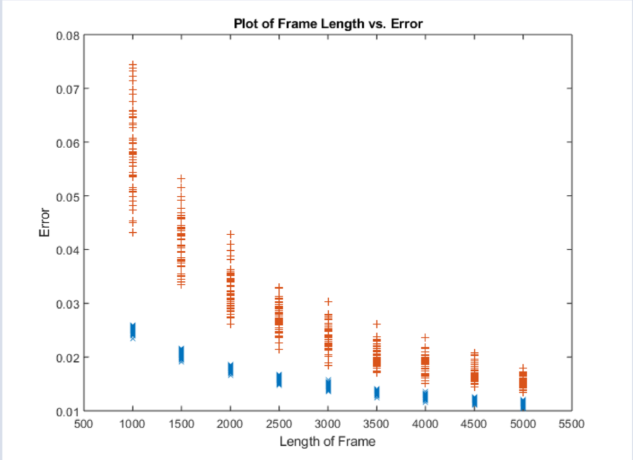
$N = 2,000$ and $|\Lambda| = 50$



$n = 250$ and $N = 1,000$







$n = 250$ and $|\Lambda| = 12$

- $+$: No reconstruction.
- $\times$: Reconstruction using Reduced Direct Inversion.

Future Direction

Our future goals include eliminating the sparsity requirement for the noise term ε in our error bound, generalizing the error bound to larger classes of frames, and performing further computer experiments to support our error bound.

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