Motivation and Goal

The goal of this project is to efficiently reconstruct signals when some of the frame measurements are erased or subject to additive noise.

Frames

A finite collection \( \{ f_j \}_{j=1}^N \) of vectors in \( \mathbb{C}^n \) is called a frame for \( \mathbb{C}^n \) if there exist positive constants \( A \) and \( B \) such that for every \( f \in \mathbb{C}^n \), \( A\|f\|^2 \leq \sum_{j=1}^N |\langle f, f_j \rangle|^2 \leq B\|f\|^2 \) holds.

- A Parseval frame is one where we can take \( A = B = 1 \).
- In finite dimensions, frames are the same thing as spanning sets.

Frame Operators

Common Operators associated with frames include the Analysis Operator \( F^* : \mathbb{C}^n \to \mathbb{C}^N \) defined as

\[
F^*f = \sum_{j=1}^N \langle f, f_j \rangle f_j.
\]

The Synthesis Operator \( F : \mathbb{C}^N \to \mathbb{C}^n \) is given by

\[
F = \left( \sum_{j=1}^N c_j^* f_j \right).
\]

The composition of the Analysis Operator and Synthesis Operator gives us the Frame Operator \( S : \mathbb{C}^n \to \mathbb{C}^n \) defined by

\[
Sf = F^* F f = \sum_{j=1}^N \langle f, f_j \rangle f_j.
\]

Reconstructions

If \( \{ f_j \}_{j=1}^N \) is a frame for \( \mathbb{C}^n \), \( S^{-1} \) exists, and for all \( f \in \mathbb{C}^n \),

\[
f = \sum_{j=1}^N \langle f, f_j \rangle f_j = \sum_{j=1}^N \langle f, S f_j \rangle f_j.
\]

The canonical dual frame to \( \{ f_j \}_{j=1}^N \) is given by \( \{ S^{-1} f_j \}_{j=1}^N \). Any frame \( \{ g_j \}_{j=1}^N \) satisfying

\[
g = \sum_{j=1}^N \langle g, f_j \rangle f_j = \sum_{j=1}^N \langle g, S f_j \rangle f_j
\]

for all \( f \in \mathbb{C}^n \) is called a dual frame to \( \{ f_j \}_{j=1}^N \). For Parseval frames, \( S = I \), and the above becomes

\[
f = \sum_{j=1}^N \langle f, f_j \rangle f_j = \sum_{j=1}^N \langle f, f_j \rangle f_j.
\]

Each of these expansions is similar to an orthonormal basis expansion, with the added bonus of redundancy.

Alice and Bob

How does Alice send the signal \( f \in \mathbb{C}^n \) to Bob given a dual frame pair \( \{ f_j \}_{j=1}^N \) and \( \{ g_j \}_{j=1}^N \)?

- Alice computes the measurements \( \langle f, g_j \rangle \).
- Alice transmits the measurements to Bob over some channel
- Bob receives the coefficients and synthesizes with the frame \( \{ f_j \}_{j=1}^N \)

In the end, Bob receives the signal \( f = \sum_{j=1}^N \langle f, g_j \rangle f_j \).

Reconstruction from Erasures

Assume that the coefficients indexed by an erasure set, \( \Lambda \), are erased in the channel. Can Bob still recover the original signal?

- If \( \{ g_j \}_{j \in \Lambda} \) is no longer a frame for \( \mathbb{C}^n \), no.
- If \( \{ g_j \}_{j \in \Lambda} \) is still a frame for \( \mathbb{C}^n \), yes.

As a general rule, as long as \( |\Lambda| < N - n \), a reconstruction is possible.

- Reduced direct inversion is an efficient method to solve this problem.

We now look at some important operators in this analysis. The Partial Reconstruction operator is defined as

\[
R_f = \sum_{j \in \Lambda} \langle f, g_j \rangle f_j.
\]

The partial reconstruction of \( f \in \mathbb{C}^n \) is thus

\[
f_R = R_f f = \sum_{j \in \Lambda} \langle f, g_j \rangle f_j.
\]

Notice that \( f_R \) is the signal Bob receives after accounting for the erasures. Notice to recover \( f \) from \( f_R \), invert \( R_c \).

\[
f = R_c^* f_R
\]

An efficient inversion formula is:

\[
f = (I + F_f G_f)^{-1} G_f ((f, g_j))_{j \in \Lambda}.
\]

where \( F_f \) is the synthesis operator for \( \{ f_j \}_{j=1}^N \), and \( G_f \) is the analysis operator for \( \{ g_j \}_{j=1}^N \).

Notice that \( R_c \) is an \( n \times n \) matrix, whereas \( (I - F_f G_f) \) is an \( |\Lambda| \times |\Lambda| \) matrix. Thus, inverting \( (I - F_f G_f) \) is more efficient.

Restricted Isometry Property

Definition: Let \( \Phi \) be an \( m \times N \) matrix, and \( s \in N \). The Restricted Isometry Constant, \( \delta_s \), of order \( s \) of \( \Phi \) is the smallest number such that

\[
(1 - \delta_s)\|x\|^2 \leq \|\Phi x\|^2 \leq (1 + \delta_s)\|x\|^2
\]

holds for all \( s \)-sparse vectors. A vector is called \( s \)-sparse if it has at most \( s \) nonzero entries.

Noise Term and Error Bounds

Let \( \Delta = I + F_f (I - G_f^* G_f)^{-1} G_f ((f, g_j))_{j \in \Lambda} \). Notice that \( \Delta \) is a linear operator, and if the measurements are subject to an additive noise term, \( \epsilon \), then the reconstructed signal is

\[
\Delta((f, g_j))_{j \in \Lambda} + \epsilon = \Delta((f, g_j))_{j \in \Lambda} + \Delta(\epsilon) = f + \Delta \epsilon.
\]

Thus, the error in the reconstruction is simply \( \Delta \epsilon \).

Theorem (Error Bounding)

Let \( \{ h_j \}_{j=1}^N \) be a Parseval frame for \( \mathbb{C}^n \). Set \( f_j = \frac{1}{\sqrt{N}} h_j \), and \( g_j = \frac{1}{\sqrt{N}} h_j \). Assume that \( |\Lambda| < N \), and \( \epsilon \) is \( s \)-sparse. Let \( \delta_s \) denote the Restricted Isometry Constant for \( \Phi \) (the synthesis operator for \( F_f \)) of order \( s \). Then,

\[
\|\Delta \epsilon\| \leq N \|\epsilon\| \frac{1 + \delta_s}{\sqrt{N}} \frac{1}{\sqrt{N}} \|\epsilon\|.
\]

Computer Experiments and Results

Goal: To better understand the relationship between the signal error, the dimension of the space, the size of the erasure set, and the length of the frame.

- The larger the size of the frame, the smaller the error.
- The larger the erasure set, the larger the error.
- For a fixed frame length, the larger the dimension, the larger the error.
- Performing the reconstruction is almost always better than not performing a reconstruction.

Future Direction

Our future goals include eliminating the sparsity requirement for the noise term \( \epsilon \) in our error bound, generalizing the error bound to larger classes of frames, and performing further computer experiments to support our error bound.

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