The Moduli Space of 3|2-dimensional Complex Associative Algebras

Tyler Gonzales, Carolyn Payne, Jory Wagner

Faculty Mentor: Michael Penkava

Department of Mathematics, University of Wisconsin–Eau Claire

University of Wisconsin Eau Claire

The Power of AND

What is a Complex Associative Algebra?

A complex associative algebra is a vector space over the complex numbers equipped with a multiplication operation which satisfies the following:

- $a * (b + c) = a * b + a * c$, (Left Distributive Law)
- $(a + b) * c = a * c + b * c$, (Right Distributive Law)
- $r(a * b) = (ra) * b = a * (rb)$ for any r in $\mathbb{C}$ (Homogeneity)
- $a * (b * c) = (a * b) * c$. (Associativity)

There are 5 basis elements of each 3|2 dimensional algebra, three even and two odd elements. Let the basis of our algebras be $\langle v_1, v_2, v_3, v_4, v_5\rangle$, where v_1 and v_2 are odd elements and v_3, v_4 and v_5 are even elements.

It is often useful to encode the multiplication structure of an algebra into a ***codifferential***. An algebra is a map $V \otimes V \to V$, and this gives rise to a coderivation of the tensor coalgebra of the parity reversion $W = \Pi(V)$.

The Fundamental Theorem of Finite Dimensional Associative Algebras

The Fundamental Theorem of Finite Dimensional Associative Algebras says that if an associative algebra structure on a finite dimensional vector space V is non nilpotent, then there is an exact sequence of algebras

$$0 \to M \to V \to W \to 0$$

where M is the *maximal nilpotent ideal* in V, and W is a *semisimple* algebra. Over the complex numbers, we have a stronger result, namely that $V = M \rtimes W$, that is, V is a semidirect product of the maximal nilpotent ideal M and a semisimple subalgebra W.

Wedderburn's Theorem classifies all simple algebras which allows us to construct all semisimple algebras. Previous constructions of moduli spaces of lower dimension give us all possible nilpotent ideals. Putting these together we can construct all non nilpotent 3|2-dimensional complex associative algebras.

For $\mathbb{Z}_2$-graded algebras, the classical theorem above still holds, but the definition of a division algebra, which arises in Wedderburn's Theorem, has to be modified as follows:

Definition. A *unital $\mathbb{Z}_2$-graded complex algebra* is a *division algebra* if every nonzero homogeneous element is invertible.

Using Theory to Construct Algebras

We build an algebra structure on a space $V = M \oplus W$ from an algebra structure μ on M and an algebra structure δ on W. The extended algebra structure d is given by

$$d = \delta + \mu + \lambda + \psi,$$

where $\lambda : M \otimes W \oplus W \otimes M \to M$ is the ***module structure*** on M
$\psi : W \otimes W \to M$ is the ***cocycle***

The associativity condition $[d, d] = 0$ is equivalent to the conditions below.

$[\mu, \lambda] = 0$, compatibility condition
$[\delta, \lambda] + \frac{1}{2}[\lambda, \lambda] + [\mu, \psi] = 0$, Maurer-Cartan condition
$[\delta + \lambda, \psi] = 0$, cocycle condition

To build new algebras, we first take generic values for λ and ψ, and then solve for the three conditions sequentially. The compatibility condition puts some constraints on the coefficients of λ as well as the fact that we are only solving up to isomorphism.

The Maurer-Cartan condition adds additional constraints on the coefficients of λ and puts some constraints on ψ.

The cocycle condition is solved to obtain our algebra structure d. We check that the new algebra d has not appeared on our list of constructed algebras so far. This process is repeated until we have constructed all of the algebras.

Deforming the Algebras

Suppose that d is an algebra, and

$$d_t = d + \psi_1 t + \psi_2 t^2 + \text{ho}$$

is a 1-parameter deformation of d, where ho stands for higher order terms in the variable t. This means that d_t is a (possibly formal) power series in t, and the associativity condition is that

$$[d_t, d_t] = 0,$$

where [,] is the *Gerstenhaber* bracket.

We say that d_t is a jump deformation of d if there is some algebra given by d' such that $d_t \sim d'$ for all $t \neq 0$ in some neighborhood of $t = 0$.

We say that d_t is a smooth deformation if $d_t \nsim d_{t'}$ for $t \neq t'$ in some neighborhood of $t = 0$.

If $\langle \delta^i \rangle$ is a basis for H^2, then there is a deformation with multiple parameters t_i, called a *versal* deformation, which encodes all of the deformations of d. The computability of a versal deformation makes it possible to determine all jumps and smooth deformations of an algebra at once. The versal deformation d^∞ is of the form

$$d^\infty = d + \delta^i t_i + \text{ho}$$

where we use the Einstein summation convention for repeated indices.

Example of a Versal Deformation

The algebra d_{285} is given by the formula $d_{285} = \psi_4^{4,4} + \psi_5^{5,5} + \psi_1^{4,1} + \psi_2^{4,2} + \psi_3^{4,3} - \psi_1^{1,5} + \psi_3^{3,4}$. Its versal deformation is given by:

$$d^\infty = \psi_4^{4,4} + \psi_5^{5,5} + \psi_1^{4,1} + \psi_2^{4,2} + \psi_3^{4,3} - \psi_1^{1,5} + \psi_3^{3,4} + t_1\psi_1^{3,1} + t_2\psi_2^{3,2}$$

We find there is a relation on the parameters t_1 and t_2 that must be satisfied in order for the expression above to give an associative algebra:

$$-2t_2^2 + 2t_1^2 = 0$$

- One solution occurs when $t_1 = t_2$. In this case, as long as $t_2 \neq 0$, we find the deformation is isomorphic to

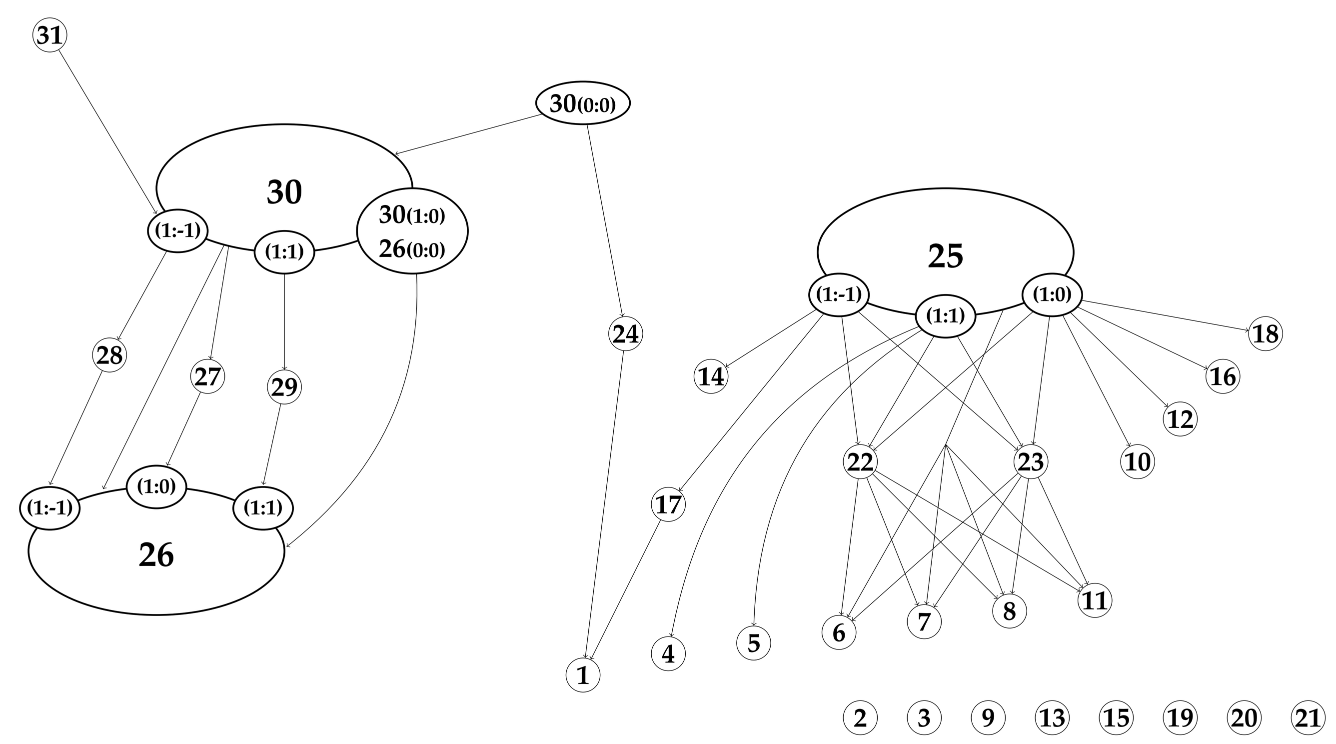
$$d_{36} = \psi_4^{4,4} + \psi_5^{5,5} + \psi_3^{3,3} + \psi_1^{3,1} + \psi_2^{3,2} - \psi_1^{1,4}$$

- Another solution occurs when $t_1 = -t_2$. In this case, again as long as $t_2 \neq 0$, we find the deformation is isomorphic to

$$d_{41} = \psi_4^{4,4} + \psi_5^{5,5} + \psi_3^{3,3} + \psi_1^{3,1} + \psi_2^{4,2} - \psi_1^{1,5}$$

Deformations of 1|3-Dimensional Algebras

This is an example of the end goal of our project. The figure below illustrates how the moduli space is glued together by deformations for 1 | 3-Dimensional Algebras.



References

[BFFPW12] L. Brunshidle, A. Fialowski, J. Frinak, M. Penkava, and D. Wackwitz. *Fundamental theorem of finite dimensional graded associative algebras.* Communications in Algebra, **40(12)**, 2012.

[FP02] A. Fialowski and M. Penkava. *Deformation theory of infinity algebras.* Journal of Algebra, **255(1)**:59–88, 2002.

[FLLPRT14] Z. Forster, H. Liu, M. Loper, M. Penkava, B. Picchiottino, A. Riedl, and T. Tran. *The moduli space of 5-dimensional non nilpotent complex associative algebras.* In preparation, 2014.

Acknowledgments

- The research for this project was conducted through the Department of Mathematics at UW–Eau Claire with funding from a student-faculty research collaboration grant and the National Science Foundation.
- Student travel funding was supported by the Office of Research and Sponsored Programs of UW–Eau Claire.
- This poster was printed by the Learning and Technology Services.