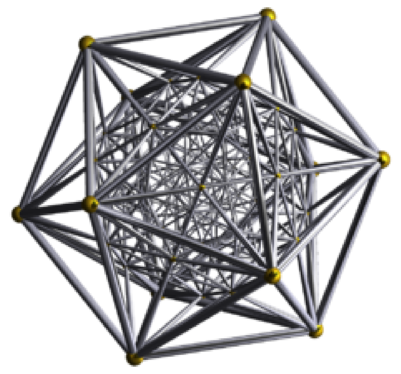

Algorithms for counting paths of fixed faces

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Motivation

There is a Hasse graph associated with each symmetry of every n-dimensional polytope, and there is an algebra associated with each Hasse graph. Each level of the graph represents the number of k-dimensional faces that remain fixed under a given automorphism (or symmetry) of the polytope. For each symmetry, we determine a polynomial $f(t)$ where the power of t represents the length of each path in the graph. The coefficient of t^0 is the number of points, the coefficient of t^1 is the number of paths of length 1, …, and the coefficient of t^i is the number of unique paths of length i in the Hasse graph. Once we determine the polynomial associated with each symmetry, we can determine the structure of the algebra associated with the symmetry using the coefficients of the Hilbert series given by the generating function

$$H(t) = \frac{t-1}{1-tf(t)}$$

Our goal is to determine the structure of all of the algebras associated with finite Coxeter groups (consisting of 4 families and 6 exceptional groups) by determining all Hasse graph polynomials $f(t)$. Duffy and past student research groups have accomplished finding the Hasse graph polynomials for the algebras associated with the A_n, B_n, D_n, $I_2(p)$ familes and H_3. We are working on the 600-Cell (H_4).

Future Directions

We next need to validate that our programs are finding what we want and explain odd anomalies, as presently we find that several variations of our brute force enumeration and graphical operation algorithms result in different polynomials for the 600-cell.

We are also looking into the idea of implementing our programs for other exceptional Coxeter groups, and as such are interested in generalizing a method to generate the conjugacy classes for any n-dimensional polytope. Regardless of whether our programs will be used, we will continue working towards finding the Hasse graph polynomials for them.

References

[1] Duffy, Colleen. Graded Traces and Irreducible Representations of $Aut(A(\Gamma))$ Acting on $grA(\Gamma)$ and $grA(\Gamma)^{-1}$. Rutger's University.

[2] Chan, Schneider, Franko. Algebra Associated with the Hasse Graph of the Hypercube. University of Wisconsin - Eau Claire. 2012.

[3] Duffy, Lemons, Riedl. n-Dimensional Semi-Hypercubes and the Algebras Associated with their Hasse Graphs. University of Wisconsin - Eau Claire, Apr. 2015.

Methods

The 600-cell, the 4-dimensional analog of a regular icosahedron, is a regular polytope with 600 tetrahedral cells, 1200 triangular faces, 720 edges, and 120 vertices. We wrote programs to: first, generate all possible symmetries. Second, determine which of each faces are fixed in each symmetry. Third, count the containment paths between each layer of fixed faces in the Hasse Graph for each symmetry. Lastly, use each containment count to generate polynomials for each unique symmetry group.

It is the stage at which containment paths are enumerated that we have recently developed different approaches. For the regular icosahedron, a brute force strategy was effective, as we were able to utilize known properties of the conjugacy classes to avoid explicitly checking every combination of potential containment. The same assumptions have not led to reliable results with the 600-cell. The task is further complicated by the fact that truly checking all possible combinations for containment is an intractable, non-polynomial time problem, such that our generalization of brute force methods used on the regular icosahedron could not complete the polynomial for one symmetry of the 600-cell after several hours of computation time. Now, we utilize alternative methods of creating specialized graph-object modules that perform built-in subgraph techniques which greatly reduce the number of combinations enumerated in time by using more memory.

Example

Procedure

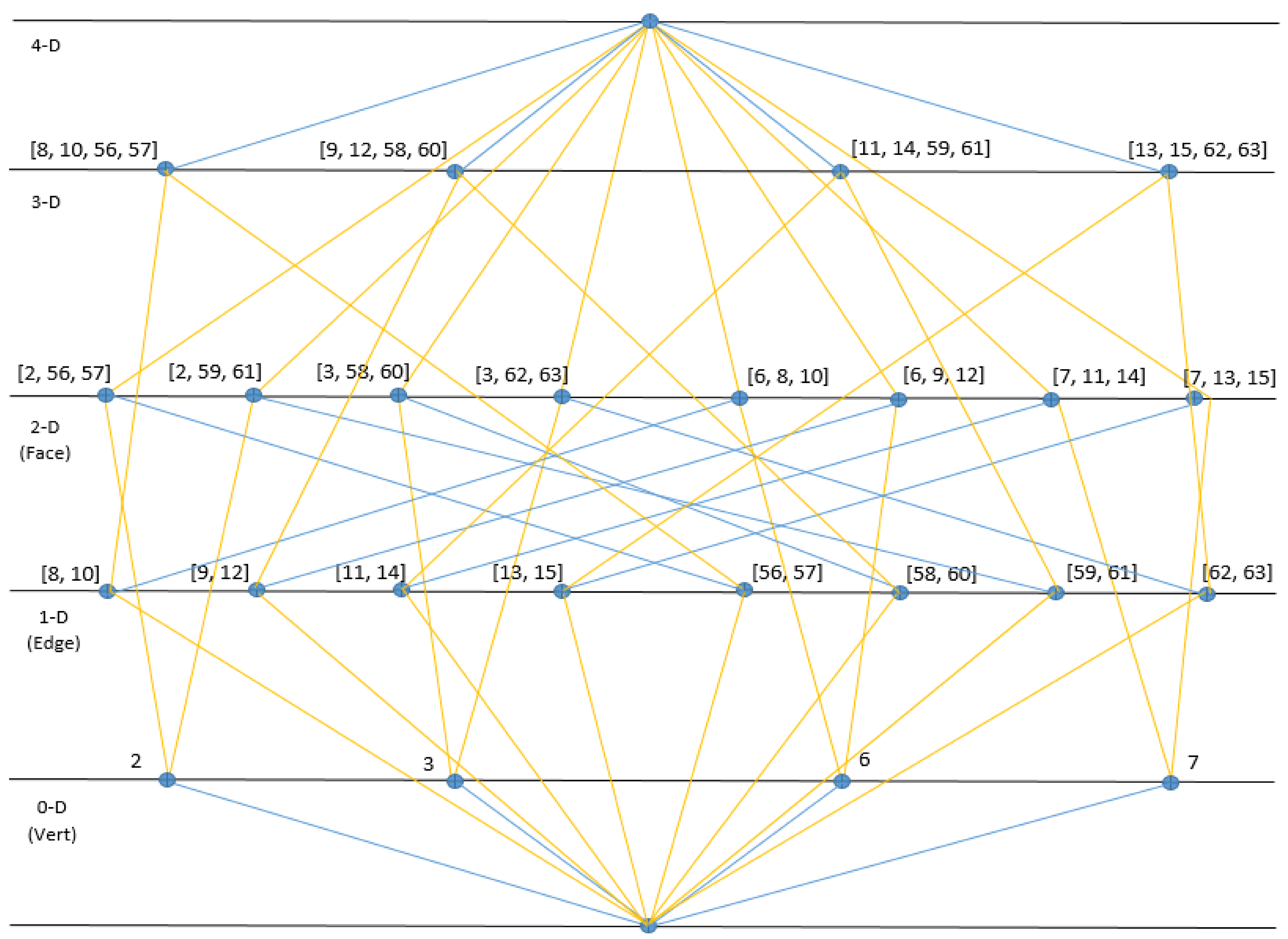
1. Apply an automorphism to the 600-cell.

$$\begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

   This symmetry will apply a reflection across the plane going through vertices 2, 3, 6 and 7

2. Count the number of faces fixed under the automorphism.
3. Construct a Hasse Graph, where the level of the graph corresponds to the dimension of the fixed face.
4. Connect the vertices in the Hasse graph to show the containments of the fixed faces.
5. Count the (signed) directed paths of each length, and create the polynomial $f(t)$ such that the coefficient of t^i is the number of directed paths of length i.

Example: reflection across plane



$$f(t) = 26 - 16t - 32t^2 + 16t^3 + 8t^4 - t^5$$

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