Self-Similarity of the 11-Regular Partition Function

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Abstract

The partition function counts the number of ways a positive integer can be written as the sum of a non-increasing sequence of positive integers. These sums are known as partitions. The famous mathematician Srinivasa Ramanujan proved the partition function has beautiful divisibility properties. We will consider the k-regular partition function, which counts the partitions where no part is divisible by k. Results on the arithmetic of k-regular partition functions have been proven by several authors. In this paper we establish self-similarity results for the 11-regular partition function.

Introduction

A *partition* of a positive integer n is a non-increasing sequence of positive integers whose sum is equal to n. The integers in a partition are called *parts*. Figure 1 shows the partitions of 4.

4
3 + 1
2 + 2
2 + 1 + 1
1 + 1 + 1 + 1
Figure 1. The partitions of $n = 4$.

The non-increasing clause in this definition precludes the creation of new partitions by reordering a sum; for example, 2 + 1 + 1 is a partition of 4 while 1 + 2 + 1 is not.

From figure 1 we see that there are five partitions of 4. The *partition function*, denoted by $p(n)$, takes as input a positive integer n and outputs the number of partitions of n. Using this notation, we have $p(4) = 5$. Table 1 shows the first few values of $p(n)$.

Table 1. First ten values of $p(n)$.

n	1	2	3	4	5	6	7	8	9	10
$p(n)$	1	2	3	5	7	11	15	22	30	42

We see from table 1 that $p(n)$ does not increase at a constant rate. Indeed, as n increases, $p(n)$ increases at a much greater rate, as seen in table 2.

Table 2. Some more values of $p(n)$.

n	10	20	30	40	50
$p(n)$	42	627	5604	37338	204226

In the early twentieth century, Srinivasa Ramanujan noticed a striking property of the partition function: on particular linear progressions, the partition function is always divisible by a specific integer. Recall that if a and b are integers, we say that a is *divisible* by b if $a = b \times k$ for some integer k. For example, 6 is divisible by 2 since $6 = 2 \times 3$, and 10 is divisible by 1, 2, 5, and 10 since $10 = 1 \times 10 = 2 \times 5$. One linear progression identified by Ramanujan is $\{4, 9, 14, 19, \ldots\}$, which is given by the formula $5n + 4$. The first several values of $5n + 4$ and $p(5n + 4)$ are given in table 3.

Table 3. $p(5n + 4)$ for $0 \le n \le 9$.

n	0	1	2	3	4	5	6	7	8	9
$5n + 4$	4	9	14	19	24	29	34	39	44	49
$p(5n + 4)$	5	30	135	490	1575	4565	12310	31185	75175	173525

Notice that each of the values of $p(5n + 4)$ ends in 0 or 5, which indicates that each is divisible by 5. In 1919, Ramanujan proved the remarkable fact that this divisibility property holds indefinitely, i.e. that $p(5n + 4)$ is divisible by 5 for all non-negative integers n. He also proved divisibilities for 7 and 11; collectively these three divisibility properties are often referred to as the *Ramanujan congruences*.

Theorem 1. The Ramanujan congruences (Ramanujan, 1921):

For all non-negative integers n,
$p(5n + 4)$ is divisible by 5,
$p(7n + 5)$ is divisible by 7,
$p(11n + 6)$ is divisible by 11.

A natural question is whether the partition function satisfies similar divisibility properties for prime numbers other than 5, 7, and 11 (recall that an integer greater than 1 is called *prime* if it is only divisible by 1 and itself). In 1967, A. O. L. Atkin and J. N. O'Brien proved that for all non-negative integers n,

$$p(157525693n + 111247) \text{ is divisible by } 13.$$

We note here that unlike the first Ramanujan congruence, this is a divisibility not readily apparent by numerical evidence. Indeed, the smallest example involves a number, namely $p(111247)$, that if written out would have over 300 digits.

A major breakthrough in this area occurred in the year 2000, when Ono proved that there is a divisibility property of this type for every prime at least 5.

Theorem 2. Ono's theorem:

For every prime $l \geq 5$, there exist positive integers a and b such that $p(an + b)$ is divisible by l for all $n \geq 0$.

Notice that in each of Ramanujan's congruences the coefficient on n matches the integer that divides the function (e.g., $p(\mathbf{5}n + 4)$ is divisible by **5**), which is not true of the Atkin and O'Brien example. Ramanujan's congruences are unique in this regard, a fact proven by S. Ahlgren and M. Boylan in 2003.

The partition function $p(n)$ is often referred to as the *unrestricted partition function*. New partition functions can be built by placing restrictions on the allowed partitions. One way to restrict partitions is by allowing no part to be divisible by a specific number. A partition is called *k-regular* if no part of the partition is divisible by k. To illustrate, consider the partition 5 + 4 of 9. This partition is not 2-regular since 4 is one of the parts, and 4 is divisible by 2. One can also verify that it is not 5-regular, and is not 4-regular. However, neither 5 nor 4 is divisible by 3, so this partition is 3-regular.

Similar to the unrestricted partition function, the k-regular partition function, denoted $b_k(n)$, counts the number of k-regular partitions of n. For instance, from figure 1 we see that the only 2-regular partitions of 4 are 3 + 1 and 1 + 1 + 1 + 1, and hence $b_2(4) = 2$. As this example suggests, $b_2(n)$ grows more slowly than $p(n)$; we illustrate this in table 4.

Table 4. Some values of $b_2(n)$ compared with $p(n)$.

n	10	20	30	40	50
$b_2(n)$	10	64	296	1113	3658
$p(n)$	42	627	5604	37338	204226

Recently, the search for Ramanujan-type congruences has been extended to the k-regular partition function. For example, in 2008 Calkin et al. proved the following congruences for b_5 and b_{13}.

Theorem 3. (Calkin, et al., 2008): For all non-negative integers n,

$b_5(20n + 13)$ is divisible by 2,
$b_{13}(9n + 7)$ is divisible by 3.

For purposes of illustration, the first few values of $b_5(20n + 13)$ are given in table 5.

Table 5. First five values of $b_5(20n + 13)$.

n	0	1	2	3	4
$20n + 13$	13	33	53	73	93
$b_5(20n + 13)$	76	5192	123500	1765642	18356026

Note that for each n in the table, $b_5(20n + 13)$ is divisible by 2. This trend continues for every $n \geq 0$.

Motivation

In recent years, a significant amount of research has been devoted to establishing *families* of Ramanujan-type congruences. In Theorem 3 we displayed a divisibility for b_{13}. In fact, Calkin et al. proved several others, which we show here.

Theorem 4. (Calkin et al., 2008): For all non-negative integers n,

$b_{13}(9n + 7)$ is divisible by 3,
$b_{13}(27n + 22)$ is divisible by 3,
$b_{13}(81n + 67)$ is divisible by 3,
$b_{13}(243n + 202)$ is divisible by 3.

Note that each subsequent linear progression can be obtained by multiplying the previous progression by 3 and adding 1; for example $3 \times (9n + 7) + 1 = 27n + 22$ and $3 \times (27n + 22) + 1 = 81n + 67$. The authors conjectured that this process can be continued indefinitely, producing an infinite family of Ramanujan-type congruences. This conjecture was proved by Webb in 2010. To better describe Webb's proof the following definition will be helpful.

Given two integers a and b, we say that a is *congruent* to b *modulo* m if $a - b$ is divisible by m. For example, 17 is congruent to 2 modulo 5 since $17 - 2 = 15$ is divisible by 5, and 4 is congruent to –3 modulo 7 since $4 - (-3) = 7$ is divisible by 7. When a is congruent to b modulo m we denote this by $a \equiv b \pmod{m}$, e.g. $17 \equiv 2 \pmod{5}$. The reader may have noticed that if we divide 5 into 17, the quotient is 3, the remainder is 2, and $17 \equiv 2 \pmod{5}$. This phenomenon is general; as another example, 3 goes into 16 five times with remainder 1, and $16 \equiv 1 \pmod{3}$. Because of this, every integer is congruent to either 0, 1, or 2 modulo 3. Note also that $a \equiv 0 \pmod{3}$ exactly when a is divisible by 3.

Returning to our discussion of $b_{13}(n)$, the key step in Webb's proof was establishing the following relationship.

Theorem 5. (Webb, 2010): For all non-negative integers n,

$b_{13}(3n + 1)$ is congruent to $-b_{13}(9n + 4)$ modulo 3.

This result can be interpreted in terms of *self-similarity*. To illustrate this notion, we display the first several values of $b_{13}(3n + 1)$ modulo 3 in the following table (recall that every integer will be congruent to either 0, 1, or 2 modulo 3).

Table 6. Values of $b_{13}(3n + 1)$ modulo 3.

n	0	**1**	2	3	**4**	5	6	**7**	8	9	**10**	11	12	**13**	14	15	**16**	17	18	**19**
$3n+1$	1	**4**	7	10	**13**	16	19	**22**	25	28	**31**	34	37	**40**	43	46	**49**	52	55	**58**
$b_{13}(3n+1)$	1	**2**	0	0	**1**	0	2	**0**	0	0	**0**	0	2	**2**	0	0	**0**	0	0	**1**

Note that the boldfaced values in the middle row of table 6 are of the form $9n + 4$. Extracting the values of $b_{13}(9n + 4)$, attaching a minus sign, and displaying them alongside the values of $b_{13}(3n + 1)$, we obtain the following:

Table 7. Comparing $b_{13}(3n + 1)$ to $-b_{13}(9n + 4)$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$b_{13}(3n+1)$	1	2	0	0	1	0	2	0	0	0	0	0	2	2	0	0	0	0	0	1
$-b_{13}(9n+4)$	-2	-1	0	0	-2	0	-1	0	0	0	0	0	-1	-1	0	0	0	0	0	-2

Since $1 \equiv -2 \pmod 3$ and $2 \equiv -1 \pmod 3$, this concurs with the statement in Theorem 5. The term *self-similarity* is used here because we have a relationship between the values of b_{13} on the linear progression $3n + 1$ and those on a progression $9n + 4$ that "sits inside" $3n + 1$.

Risking an analogy, picture a set of Russian nesting dolls, where the largest is black, the next largest is white, the next black, and so on. In a similar fashion, the sequence $b_{13}(9n + 4)$ "lives inside" $b_{13}(3n + 1)$ as its negative ("white inside black"). Note that if we were to repeat this process, we would have

$$b_{13}(27n + 13) \equiv -b_{13}(9n + 4) \equiv -(-b_{13}(3n + 1)) \equiv b_{13}(3n + 1) \pmod 3$$

(and we're back to black).

Once the self-similarity result in Theorem 5 was established, Webb obtained the promised infinite family of Ramanujan-type congruences in the following way. Recall that Calkin et al. proved that $b_{13}(9n + 7)$ is divisible by 3, i.e. $b_{13}(9n + 7) \equiv 0 \pmod 3$. Next, note that $9n + 7$ has the form $3k + 1$ since $3(3n + 2) + 1 = 9n + 7$. This means that one can apply the result from Theorem 5 to conclude that

$$b_{13}(9n+7) \equiv b_{13}(3(3n+2)+1) \equiv -b_{13}(9(3n+2)+4) \equiv -b_{13}(27n+22) \pmod 3.$$

Since $b_{13}(9n + 7) \equiv 0 \pmod 3$, it follows that $b_{13}(27n + 22) \equiv 0 \pmod 3$. This process can be repeated as many times as we wish, yielding infinitely many divisibilities (including those in Theorem 4).

The focus of our research was to establish self-similarity results analogous to that in Theorem 5. We now state our main result.

Theorem. For all non-negative integers n,

$$b_{11}(125n + 10) \equiv b_{11}(3125n + 260) \pmod 5,$$
$$b_{11}(125n + 35) \equiv b_{11}(3125n + 885) \pmod 5,$$
$$b_{11}(125n + 110) \equiv b_{11}(3125n + 2760) \pmod 5.$$

In the next section we give an informal overview of our research methods, and in the final section we provide a proof of our theorem.

Methods

Recall that Webb's argument for the infinite family of Ramanujan-type congruences for b_{13} had two key ingredients: the congruence $b_{13}(9n + 7) \equiv 0 \pmod 3$, which provides a beginning, and the self-similarity result in Theorem 5, which enables one to use a single congruence to create more. Because of this, the first step was to identify potential congruences and self-similarities. We illustrate the former using the example

$b_{13}(9n+7) \equiv 0 \pmod 3$. Begin by calculating $b_{13}(n)$ modulo 3 for all non-negative integers n within a specified range. We display the first 35 values here.

$$1, 1, 2, 0, 2, 1, 2, 0, 1, 0, 0, 2, 2, 1, 2, 0, 0, 1, 0, 2, 0, 2, 0, 1, 1, 0, 0, 0, 0, 2, 1, 0, 0, 0, 0$$

Next a computer script is used to search through this list of values for linear progressions that match the desired results. For example, the progression $9n + 5$ picks out the following boldfaced values in the sequence:

$$1, 1, 2, 0, 2, \mathbf{1}, 2, 0, 1, 0, 0, 2, 2, 1, \mathbf{2}, 0, 0, 1, 0, 2, 0, 2, 0, \mathbf{1}, 1, 0, 0, 0, 0, 2, 1, 0, \mathbf{0}, 0, 0$$

Since this progression yields several values that are not congruent to 0 modulo 3, the script rejects it and moves on to the next linear progression. When the script arrives at $9n + 7$ it finds

$$1, 1, 2, 0, 2, 1, 2, \mathbf{0}, 1, 0, 0, 2, 2, 1, 2, 0, \mathbf{0}, 1, 0, 2, 0, 2, 0, 1, 1, \mathbf{0}, 0, 0, 0, 2, 1, 0, 0, 0, \mathbf{0}, 0,$$

and tags this as a potential congruence since each term of the sequence is congruent to 0 modulo 3. Of course, we cannot test infinitely many values of $b_{13}(9n + 7)$, and therefore this congruence cannot be verified by computation alone.

To overcome this obstacle, we use the theory of *modular forms*. Over the past 25 years, modular forms have taken a prominent place in number theory. This is due in part to the major role they played in Wiles's epochal proof of Fermat's Last Theorem. Modular forms have three key properties that are critical to our work. First, every modular form has a Fourier expansion, i.e. an infinite sum of the form

$$a(0) + a(1)q + a(2)q^2 + a(3)q^3 + a(4)q^4 + \cdots,$$

where $a(n)$ is a function that takes nonnegative integers as input, and outputs complex numbers. As a shorthand, we will denote this sum by

$$\sum_{n=0}^{\infty} a(n)q^n.$$

Second, in some cases we can find modular forms whose Fourier coefficients (the $a(n)$ function) carry number theoretic information. One example of this phenomenon is the weight 4 normalized Eisenstein series, which is defined by

$$E_4 := 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n.$$

The coefficients in this series involve the function $\sigma_3(n)$, which sums the cubes of the divisors of n. If modular forms can be constructed so that our linear progressions are encoded within the coefficients, then we can bring the theory to bear. The third key property of modular forms is that they are very rare, which brings us to a vital result—Sturm's Theorem. The theorem states that if two modular forms are congruent modulo *m* up to a certain bound, they are congruent modulo *m* everywhere.

In order to relate the *k*-regular partition function to modular forms, generating functions will be used. For example, the infinite sum

$$p(0) + p(1)q + p(2)q^2 + p(3)q^3 + p(4)q^4 + \cdots = \sum_{n=0}^{\infty} p(n)q^n$$

is a generating function for $p(n)$ (we adopt the convention that $p(0) = b_k(0) = 1$). Note that this sum is simply another way to package the values of $p(n)$, but this change in viewpoint is extremely powerful, as we may think of this object as a single function of q. This generating function satisfies the identity

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)},$$

where the notation on the right-hand side is shorthand for the infinite product

$$\frac{1}{(1-q)} \times \frac{1}{(1-q^2)} \times \frac{1}{(1-q^3)} \times \frac{1}{(1-q^4)} \times \frac{1}{(1-q^5)} \times \cdots.$$

As we will see in the next section, this expression allows us to relate the generating function of $b_{11}(n)$ to modular forms.

Results

As mentioned previously, our main result is a set of analogues of the self-similarity result by Webb in Theorem 5.

Theorem. For all nonnegative integers n,

$$\begin{aligned} b_{11}(125n + 10) &\equiv b_{11}(3125n + 260) \pmod 5, \\ b_{11}(125n + 35) &\equiv b_{11}(3125n + 885) \pmod 5, \\ b_{11}(125n + 110) &\equiv b_{11}(3125n + 2760) \pmod 5. \end{aligned}$$

Proof: The generating function for $b_k(n)$ satisfies the identity

$$\sum_{n=0}^{\infty} b_k(n)\, q^n = \prod_{n=1}^{\infty} \frac{(1-q^{kn})}{(1-q^n)}.$$

In order to use Sturm's Theorem we must relate our *k*-regular partition functions to modular forms. To that end, we recall Dedekind's eta function

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n),$$

where $q = e^{2\pi iz}$, and z is a complex variable. We will write $z = x + iy$, where x and y are real numbers. Note the resemblance to the generating function of $b_k(n)$; remarkably, one can also use $\eta(z)$ to build modular forms.

We now provide some background on modular forms. Details will be omitted here in favor of illustrating some of these concepts as they arise in our proof. Given a positive integer N, let

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \text{ c} \equiv 0 \text{ (mod } N) \right\}.$$

Let $\mathbb{H} := \{z \in \mathbb{C} \mid y > 0\}$, and for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $z \in \mathbb{H}$ define $\gamma z := \frac{az+b}{cz+d}$. Suppose k is a positive integer, χ is a Dirichlet character modulo N, and $f: \mathbb{H} \to \mathbb{C}$ is holomorphic. Then f is said to be a *modular form* of weight k on $\Gamma_0(N)$ with character χ if

$$f(\gamma z) = \chi(d)(cz+d)^k f(z)$$

for all $\gamma \in \Gamma_0(N)$ and f is holomorphic at the cusps of $\Gamma_0(N)$. Using standard notation, we denote the complex vector space of modular forms of weight k on $\Gamma_0(N)$ with character χ by

$$M_k(\Gamma_0(N), \chi).$$

We use χ_d to denote the character $\chi_d(n) = \left(\frac{d}{n}\right)$.

Recall the weight 4 normalized Eisenstein series defined previously. This is a modular form on $\Gamma_0(1)$ with trivial character. Since 240 is divisible by 5, it follows that $240 \equiv 0 \pmod 5$. Hence, if we consider E_4 modulo 5, we have that $E_4 \equiv 1 \pmod 5$. Thus multiplying a modular form by powers of E_4 does not change the form when considered modulo 5.

As mentioned previously, Sturm's Theorem provides a method to test whether two modular forms are congruent. Let us make this more explicit. If $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in \mathbb{Z}[[q]]$ and m is a positive integer, let $\text{ord}_m(f(z))$ be the smallest n for which $a(n) \not\equiv 0 \pmod m$ (if there is no such n, we define $\text{ord}_m(f(z)) = \infty$).

Sturm's Theorem: Suppose m is prime and $f(z), g(z) \in M_k(\Gamma_0(N), \chi) \cap \mathbb{Z}[[q]]$. If

$$\text{ord}_m\big(f(z) - g(z)\big) > \frac{k}{12}[SL_2(\mathbb{Z}) : \Gamma_0(N)],$$

then $f(z) \equiv g(z) \pmod m$, i.e., $\text{ord}_m(f(z) - g(z)) = \infty$.

Note that $[SL_2(\mathbb{Z}) : \Gamma_0(N)] = N \cdot \prod \frac{l+1}{l}$, where the product is over the prime divisors of N. We will refer to the number $\frac{k}{12}[SL_2(\mathbb{Z}) : \Gamma_0(N)]$ as the *Sturm bound*.

We now define the concept of *Hecke operators*. If $f(z) = \sum_{n=0}^{\infty} a(n)q^n$ and m is prime, then the action of the *Hecke operator* $T_{m,k,\chi}$ on $f(z)$ is defined by

$$(f|T_{m,k,\chi})(z) := \sum_{n=0}^{\infty}\left(a(mn) + \chi(m)m^{k-1}a(n/m)\right)q^n.$$

Note that if $k > 1$, then

$$(f|T_{m,k,\chi})(z) \equiv \sum_{n=0}^{\infty} a(mn)q^n \pmod m.$$

We will use the abbreviation $T_m := T_{m,k,\chi}$. The *Hecke operator* is a linear transformation on $M_k(\Gamma_0(N), \chi)$. Note that T_m picks out every m^{th} coefficient of f modulo m. Applying T_m to our modular forms l times will allow us to isolate linear progressions of the form $b_k(m^l n + c)$.

For the sake of brevity, we will only prove the first congruence stated in the theorem. We begin by constructing the functions

$$g_1(z) := \frac{\eta(11z)}{\eta(z)}\eta^{2750}(z)E_4^{8250}(z)$$

and

$$g_2(z) := \frac{\eta(11z)}{\eta(z)}\eta^{68750}(z).$$

By standard criteria on eta quotients (Ono, 2004), both g_1 and g_2 are modular forms in the space $M_{34375}(\Gamma_0(11), \chi_{-11})$. Considering g_1 modulo 5 and using the previous identity for the generating function of $b_k(n)$, we have that

$$g_1(z) \equiv \sum_{n=0}^{\infty} b_{11}(n)q^{n+115} \cdot \prod_{n=1}^{\infty}(1-q^n)^{2750} \pmod 5$$

and

$$g_2(z) = \sum_{n=0}^{\infty} b_{11}(n)q^{n+2865} \cdot \prod_{n=1}^{\infty}(1-q^n)^{68750}.$$

The first few terms of $g_1(z)$ modulo 5 are as follows:

$$q^{115} + q^{116} + 2q^{117} + 3q^{118} + 2q^{120} + q^{121} + 2q^{123} + 2q^{125} + q^{127} + 4q^{128} + 2q^{129} + \cdots$$

Since the coefficients of q^{115}, q^{120}, and q^{125} are 1, 2, and 2 respectively, and recalling that the Hecke operator T_5 picks out every fifth coefficient of the series, applying T_5 to $g_1(z)$ yields the following:

$$(g_1|T_5)(z) \equiv q^{23} + 2q^{24} + 2q^{25} + q^{26} + \cdots \pmod 5)$$

In product form this sum can be written

$$(g_1|T_5)(z) \equiv \sum_{n=0}^{\infty} b_{11}(5n)q^{n+23} \cdot \prod_{n=1}^{\infty}(1-q^n)^{550} \pmod 5.$$

By the same reasoning, applying T_5 three times to g_1 and five times to g_2 yields

$$(g_1|T_5^3)(z) \equiv \sum_{n=0}^{\infty} b_{11}(125n+10)q^{n+1} \cdot \prod_{n=1}^{\infty}(1-q^n)^{22} \pmod 5$$

and

$$(g_2|T_5^5)(z) \equiv \sum_{n=0}^{\infty} b_{11}(3125n+260)q^{n+1} \cdot \prod_{n=1}^{\infty}(1-q^n)^{22} \pmod 5.$$

Since the weight of our modular forms is $k = 34275$ and the level is the prime $N = 11$, the Sturm bound is

$$\frac{34275}{12} \cdot 11 \cdot \frac{11+1}{11} = 34275.$$

A computation verifies that the first 34275 coefficients of these two series are congruent modulo 5. Hence, by Sturm's Theorem, $(g_1|T_5^3)(z) \equiv (g_2|T_5^5)(z) \pmod 5$. Cancelling the common factor of $\prod_{n=1}^{\infty}(1 - q^n)^{22}$, we conclude that

$$b_{11}(125n+10) \equiv b_{11}(3125n+260) \pmod 5$$

for all $n \geq 0$. ■

Conclusion

A significant amount of research has been devoted to proving Ramanujan-type congruences for the unrestricted partition function and the k-regular partition function. Self-similarity results have been established by several authors for $b_k(n)$, and these have been used to construct infinite families of congruences. Currently, we are researching possible base cases that, together with our self-similarity results, could lead to new families of congruences. There is no doubt that future work will continue to explore the beautiful patterns and mysteries of partitions.

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