Analysis of Combining Musical Scale Vectors

Emily Gullerud | Dr. James Walker and Dr. Gary Don | Departments of Mathematics and Music, University of Wisconsin-Eau Claire

MUSICAL EXAMPLES

1. Rebecca Clarke, Sonata for Viola & Piano, 2nd movement
   - The octatonic scale as a combination of two major triads a tritone apart
   - Missing two of the eight notes of the octatonic scale
   The combination according to the mathematical methods:
     \[ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{*the octatonic is a weighted combination of all 12 major triads} \]
     \[ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{*the octatonic is a combination of two major triads a tritone apart} \]

2. Béla Bartók, “From the Isle of Bali”, No. 109
   - The octatonic scale as a combination of four minor seconds
   - Emphasis on the half steps
   The combination according to the mathematical methods is the same as in this piece of music:
     \[ \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{*the octatonic is a combination of four major thirds two semitones apart} \]

3. Claude Debussy, Preludes II: Voiles
   - The whole tone scale as a combination of four major thirds
   - The entire piece uses the whole tone scale, with the exception of one portion using the pentatonic scale, though the beginning emphasizes the decomposition into major thirds
   The combination according to the mathematical methods:
     \[ \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \quad \text{*the whole tone scale is a weighted combination of six major thirds two semitones apart} \]

4. Rimsky-Korsakov, Sadko
   - The octatonic scale as a combination of four dominant seventh chords
   - The roots of the four dominant seventh chords form a diminished seventh chord
   The combination according to the mathematical methods:
     \[ \begin{pmatrix} -1 & 3 & 5 & 7 \end{pmatrix} \quad \text{*the octatonic is a weighted combination of all 12 dominant seventh chords} \]

METHODOLOG

Step 1: Create a circulant matrix
- Take the scale vector for the note collection which will make up the linear combination
- Set the scale vector as the first column of a \( 12 \times 12 \) matrix
- Each successive column is the scale vector rotated one slot
- Example: interval of a major third

Step 2: Set up the matrix equation
- Let \( b \) be the scale vector for a different note collection than the circulant matrix.
- Let \( z \) be the coefficients necessary to represent \( b \) in terms of the columns of \( A \).
- Then \( Ax = b \)

Step 3: Use the Pseudoinverse
- Solve for \( x \) using the pseudoinverse of \( A \). Example: An octatonic scale as a combination of diminished seventh chords:
  \[ x = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \]
  Iterative method: \( \text{tfqmr} \)

Step 3: The circulant matrix is singular
- Find the new matrix into reduced row echelon form
- Decompose into a \( 12 \times 12 \) matrix \( A' \) and a \( 12 \times 1 \) matrix \( b' \)
- Solve \( A'x = b' \) using the Transpose-free quasi-minimal residual iterative method

REFERENCES