Motivation

There is a Hasse graph associated with each symmetry of every n-dimensional polytope, and there is an algebra associated with each Hasse graph. Each level of the graph represents the number of k-dimensional faces that remain fixed under a given automorphism (or symmetry) of the polytope. For each symmetry, we determine a polynomial \( f(t) \) where the power of \( t \) represents the length of each path in the graph. The coefficient of \( t^i \) is the number of points, the coefficient of \( t^i \) is the number of paths of length 1, ..., and the coefficient of \( t^i \) is the number of unique paths of length \( i \) in the Hasse graph. Once we determine the polynomial associated with each symmetry, we can determine the structure of the algebra associated with the symmetry using the coefficients of the Hilbert series given by the generating function \( H(t) = \frac{f(t)}{1 - t} \).

Our goal is to determine the structure of all of the algebras associated with finite Coxeter groups (consisting of 4 families and 6 exceptional groups) by determining all Hasse graph polynomials \( f(t) \). Duffy and past student research groups have accomplished finding the Hasse graph polynomials for the algebras associated with the \( A_n, B_n, D_n, I_2(p) \) families and \( H_3 \). We are working on the 600-Cell \((H_4)\).

Methodology

We determined the best way to represent the vertices of the 600-cell and created a program in Java to determine the sets of faces. From there, we wrote Maple programs which do the following:

- One creates a symmetry matrix for each symmetry by looking at where a given 3-D face (flag) gets mapped to. Another identifies a representative for each symmetry class.
- We have one program for each dimension to determine which faces are fixed by creating a matrix formed by the vertices for each face and multiplying that by a representative of each symmetry class.

Throughout this process, we have been using our programs on the icosahedron to compare and verify that our results are accurate. Our next step will involve using our programs to determine the containments of the fixed k-dimensional faces in order to create the Hasse graphs.

Example

**Procedure**

1. Apply an automorphism to the 600-cell.
   
   \[
   \begin{pmatrix}
   -1 & 0 & 0 & 0 \\
   0 & 1 & 0 & 0 \\
   0 & 0 & -1 & 0 \\
   0 & 0 & 0 & 1 
   \end{pmatrix}
   \]

   This symmetry will apply a reflection across the plane going through vertices 2, 3, 6 and 7

2. Count the number of faces fixed under the automorphism. Running it through our programs, we get the faces shown in the diagram.

3. Using the fixed faces, construct a Hasse Graph, where the level of the graph corresponds to the dimension of the fixed face.

4. Connect the points in the Hasse graph to show the containments of the fixed faces.

5. Count the (signed) directed paths of each length, and create the polynomial \( f(t) \) such that the coefficient of \( t^i \) is the number of directed paths of length \( i \).

\[
f(t) = 26 - 16t - 32t^2 + 16t^3 + 8t^4 - t^5
\]

References