1.2 Splines

A spline function is a piecewise defined function with certain smoothness conditions [Cheney]. A wide variety of functions is potentially possible; polynomial functions are almost exclusively used. The most often used form is the cubic spline. One part of the piecewise defined function is shown in (Eqn. 1.2-1). The parameters $a_{i,0}$ through $a_{i,3}$ are not known yet. Four equations are necessary to evaluate these coefficients.

$$f_i(x) = a_{i,0} + a_{i,1} \cdot x + a_{i,2} \cdot x^2 + a_{i,3} \cdot x^3$$

Eqn. 1.2-1

There are two sorts of Splines, **ordinary Splines** and **B-Splines**. The two spline functions have the same general structure regarding the piecewise defined function and smoothness conditions. The difference is that ordinary Splines go through all the data points ***exactly*** whereas B-Splines do not necessarily fit the data exactly.

Splines, especially B-Splines, are well known and very powerful tools to perform curve fits [Cheney]. Often they are preferred over many other methods. Sometimes the preference is justified although sometimes other methods would perform just as good or even better but maybe less effort would be needed. A discussion describing the procedure and its limitations, as well the advantages over other methods seemed therefore necessary.

1.2.1 Ordinary Splines

1.2.1.1 Algorithm

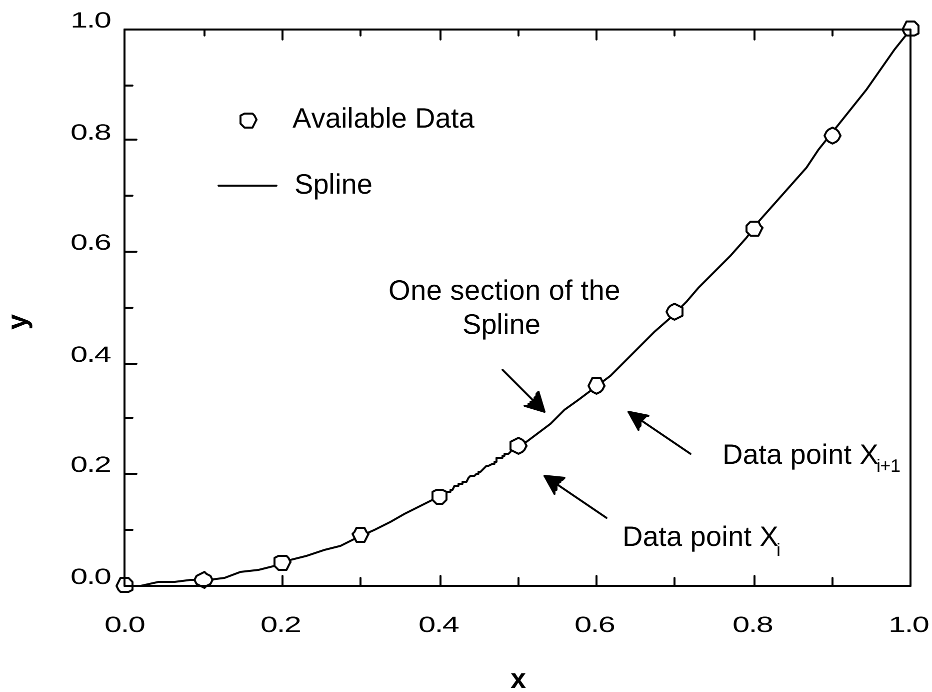

Fig. 1.2-1 Result for a ordinary Spline Function

For ordinary Splines the curve has to go through all the points (Fig. 1.2-1), therefore (Eqn. 1.2-2)

$$f(x_i) = y_i$$

Eqn. 1.2-2

has to be satisfied for all the points ($x_i|y_i$). The spline function $f(x)$ has to yield the value y_i for x_i. In addition to (Eqn. 1.2-2) the smoothness conditions have to be fulfilled. The smoothness conditions require that all of the derivatives and the function value of each section of the function have to be equal at the points where the Splines join (Eqn. 1.2-2). In this example it is shown for the pieces of the spline joining at the available data points. For a cubic spline these smoothness conditions require that the slope, the curvature and the third derivative of each piece of the spline function has to be equal to the slope, the curvature and the third derivative of the neighboring piece at the point where those pieces join.

$$\boldsymbol{f_j^k}(X_i) = \boldsymbol{f_{j-1}^k}(X_i)$$

Eqn. 1.2-3

This means for a spline of order k with n sections and $n+1$ data points, that there are $(k+1)n$ unknowns. Since there are $(n+1)$ points that the spline has to pass, the number of equations is: $(n+1)$ equations for passing through the available data (Eqn. 1.2-2) and $(k+1)n$ smoothness conditions (Eqn. 1.2-2). The degrees of freedom, the difference between the unknowns and the number of equations, therefore is $(k+1)n - (n+1) - k(n-1) = k-1$. A system of equations with any degree of freedom cannot be solved exactly. k-1 more equations need to be found in order to solve this problem. These equations can be arbitrary conditions or they can reflect characteristics of the data. These characteristics can for example be knowledge that results out of physical relationships that are valid for only certain ranges of data or for idealized conditions.

Spline functions are not limited to problems with only one independent variable and they can be expanded to more dimensional problems. This implies as well that there will be more degrees of freedom and therefore more additional conditions are necessary.

1.2.1.2 Results and Limitations

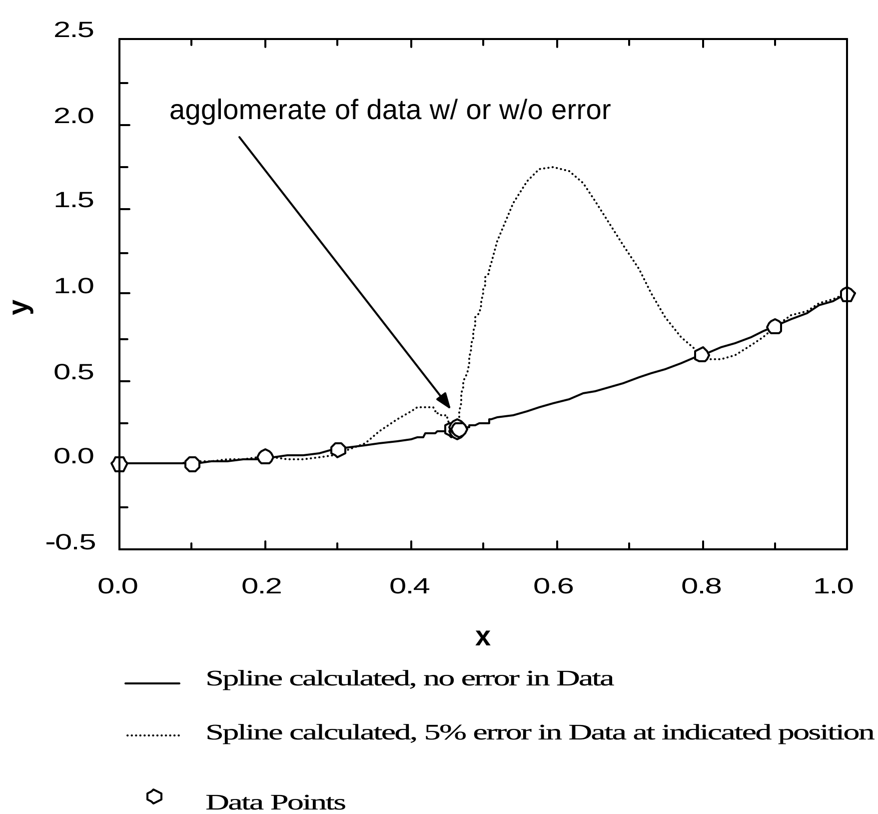

Fig. 1.2-2 Comparison of ordinary Spline for slightly noisy data compared to non-noisy data

A problem with ordinary spline functions is that the function is forced fit the data exactly. In general it can be assumed that the data include error. An extreme example is shown in (Fig. 1.2-2). Three points that including *5%* error are very close together. The spline can still fit this problem but the function values of the spline usually deviate significantly from the expected (Fig. 1.2-2).

1.2.2 B-Splines

1.2.2.1 Algorithm

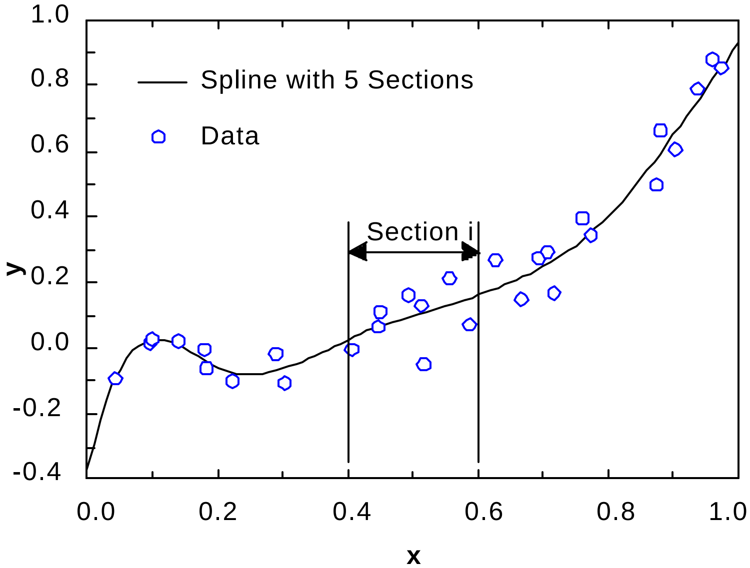

Fig. 1.2-3 general B-spline

B-Splines belong to the same family as the ordinary Splines. They are piecewise defined functions, usually polynomials. B-Splines have the same smoothness conditions as ordinary Splines (Eqn. 1.2-2). B-Splines are not forced through the datapoints exactly; the function has simply to come close to the datapoints. This means that (Eqn. 1.2-2) is not used for B-Splines. There are $(k+1)n$ unknown parameters for a spline function of order k that uses n sections. Only the smoothness conditions are available as equations to equate the unknowns. With $k(n-1)$ smoothness conditions this leaves $n+k$ degrees of freedom. In order to evaluate all the values for the coefficients it would be necessary to include $n+k$ more equations. Another way to evaluate the unknown coefficients is to adjust the coefficients such that they minimize the Sum of Squares (Eqn. 1.1-3). This method of minimizing the Sum of Squares was discussed in Section 1.1. For this problem the minimization has to be performed for $n+k$ parameters. The minimization process can be simplified by including more relations that are known about the data. For each independent information added, the degree of freedom will decrease by one.

1.2.2.2 Results and Limitations

B-Splines have the advantage over ordinary Splines that the sensitivity decreases as more data points are in each section. But if there are too many datapoints in each section the interpolation worsens. The interpolation can turn into a curve that is far of all the points, does not even follow any trend but still represents the minimum of the Sum of Squares (Fig. 1.2-4). In (Fig. 1.2-4) an extreme example of a cubic B-Spline with one section is shown. The data and the spline deviate heavily. The effect is heavily exaggerated here but slighter effects of this problem are possible [Cheney].

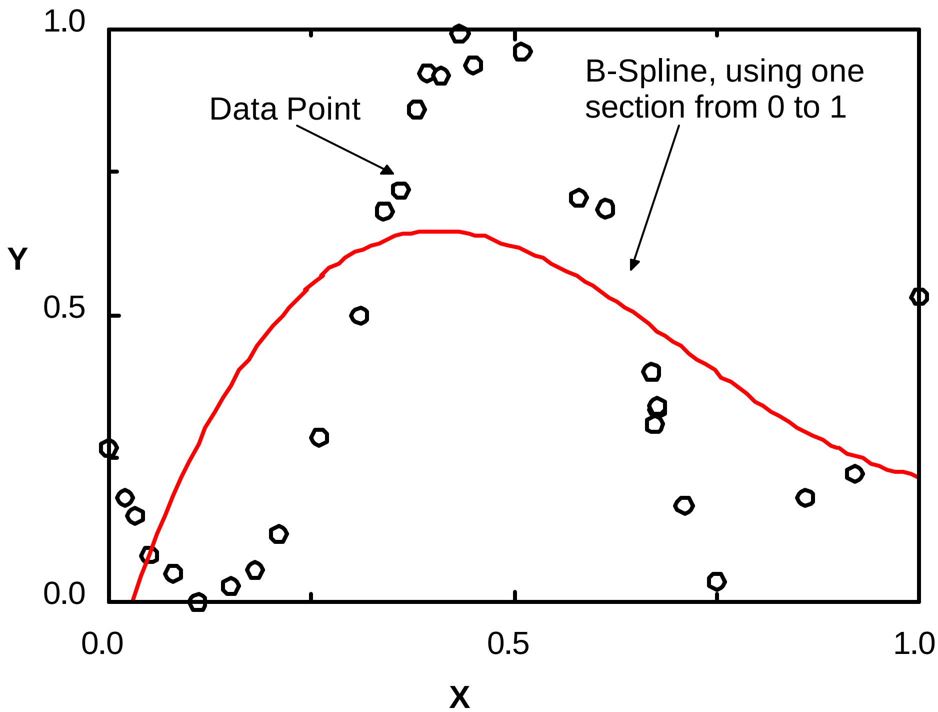

Fig. 1.2-4 B-Spline with not enough sections

Arbitrarily chosen conditions can have an unpredictable impact on B-Splines. In (Fig. 1.2-5) several results of B-Splines are shown. They all follow the trend of the data well. But they are able to introduce some effects to the function that can cause problems depending on the later use of the results.

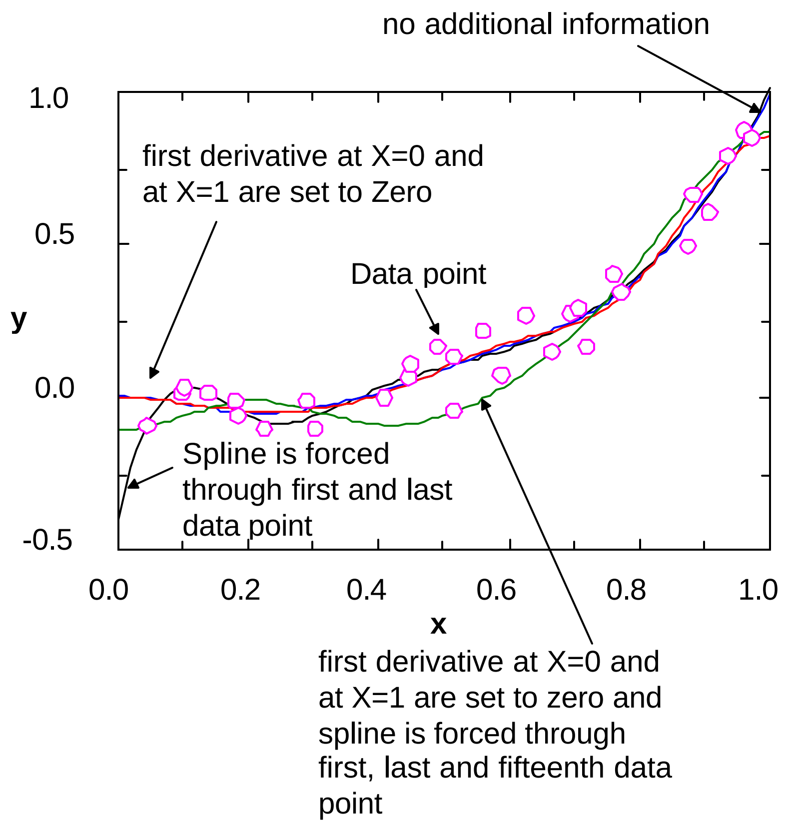

Fig. 1.2-5 B-Spline including different additional information and not