Determinants of Rational Tangles and Their Closures

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Introduction

Knot theory is a field of topology that studies the embedding of a circle in $\mathbb{R}^3$. In various biological processes strands of DNA can be represented with knots. In these instances it is useful to view DNA strands topologically and model the operations mathematically to get a better understanding of what the enzymes involved in these processes are doing. The actions of these enzymes, called topoisomerases, on DNA can be represented with topological operations. This is helpful to molecular biologists in analyzing DNA and working on it to solve genetics related issues and cure genetic diseases.

Our research team studied rational tangles, which can be thought of as two tangled strings with fixed ends. We researched the colorability of rational tangles. Colorability is an invariant of a tangle, so it is very helpful to understand more about it.

Definitions

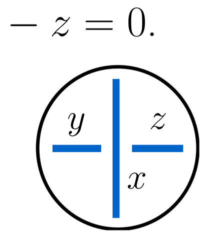
Link- A collection of curves in $\mathbb{R}^3$ that are non-intersecting and closed. A knot is a link of one component.

Rational tangle- A rational tangle is comprised of 2 strings with free ends, which can be deformed to make two straight lines if the endpoints of one string are held in a fixed position on the outer surface of a sphere and the endpoints of the second string are manipulated around the first string. A 3-component tangle is denoted (m, n, p) where $m, n, p \in \mathbb{Z}$, this can be visualized by the first figure in the Closure of Tangles section, m is the number of twists in the first component, n is the number of twists in the second component, etc.

Tangle fraction- A complete invariant that is a continued fraction dealing with the number of components in a tangle. Reducing this fraction will give $\mathbb{Q}$. In a n-component tangle the tangle fraction will be

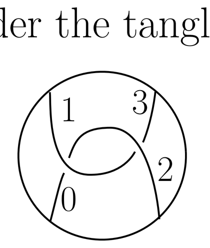
Method of Coloring a Tangle

Each intersection of a tangle must follow the same equation as the knot intersections, $2x - y - z = 0$.



$$C_T = \begin{bmatrix} NE & NW \\ SE & SW \end{bmatrix}$$

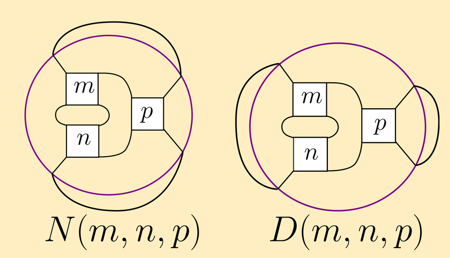
Let the upper northeast corner strand of the tangle to be 1. Then following the equation of the crossing, assign the remaining curves accordingly to satisfy the equation $2x-y-z = 0$. For the coloring of the tangle we are only interested in the numbers associated with the curves that are on the border of the circle that encompasses the tangle, depicted below. These values make up the Coloring matrix of the Tangle, denoted C_T. The prime factors of $|\det(C_T)|$ will help us determine the colorability of the tangle when closed. Consider the tangle (2,0,0) which would be thought of as



$$C_{(2,0,0)} = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \qquad \det(C_{(2,0,0)}) = 2$$

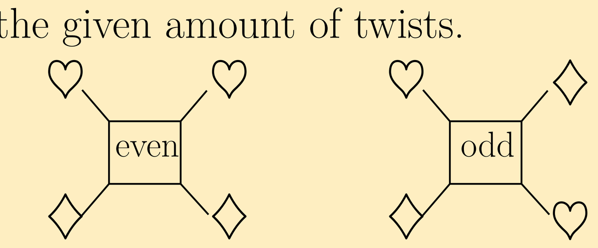
Closures of Tangles

Numerator closure for a (m, n, p) tangle.



Denominator closure for a (m, n, p) tangle.

Closing the tangle will result in a knot, K, or link, L, depending on the values of m, n, p. Note if a given component is even the NW strand is the same as the NE strand. Conversely, if a given component is odd the NW strand is the same as the SE strand after the given amount of twists.



Following the strands it can easily be seen when the given (m, n, p) tangle is a knot or link given the numerator closure, N and the denominator closure, D.

m	n	p	N	D
even	even	even	L	K
even	even	odd	K	K
even	odd	even	L	K
even	odd	odd	K	K
odd	even	even	K	K
odd	even	odd	L	K
odd	odd	even	K	L
odd	odd	odd	K	L

Theorem: The tangle fraction for a rational tangle is $\frac{a_n}{b_n} = \frac{t_n a_{n-1} + b_{n-1}}{a_{n-1}}$

pf: Let T_n be a rational tangle with n-components, s.t. $n \in \mathbb{N}$. The set $\{t_1, t_2, t_3, ..., t_n\}$ represents the number of twists in each component,

The initial condition is when $n = 1$, the tangle fraction is $\frac{t_1}{1}$

Base Case: When $n = 2$ the tangle fraction is

$$t_2 + \frac{1}{t_1} = \frac{t_1 t_2 + 1}{t_1} = \frac{a_1 t_2 + 1}{a_1} = \frac{a_2}{b_2} \quad \checkmark$$

Inductive Case: [WTS: $\frac{a_{n+1}}{b_{n+1}} = \frac{t_{n+1} a_n + b_n}{a_n}$] Assume the following statement is true for all m s.t. $1 \leq m \leq n$ $t_n + \cfrac{1}{t_{n-1} + \cfrac{1}{\ddots\, t_2 + \cfrac{1}{t_1}}} = \frac{a_n}{b_n}$

By definition the tangle fraction for a tangle with $n+1$-components will be

$$t_{n+1} + \cfrac{1}{t_n + \cfrac{1}{t_{n-1} + \cfrac{1}{\ddots\, t_2 + \cfrac{1}{t_1}}}} = \frac{a_{n+1}}{b_{n+1}}$$

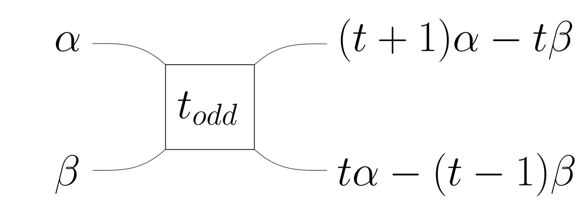
$$= t_{n+1} + \cfrac{1}{\frac{a_n}{b_n}} \quad \text{by inductive hypothesis}$$

$$= t_{n+1} + \frac{b_n}{a_n}$$

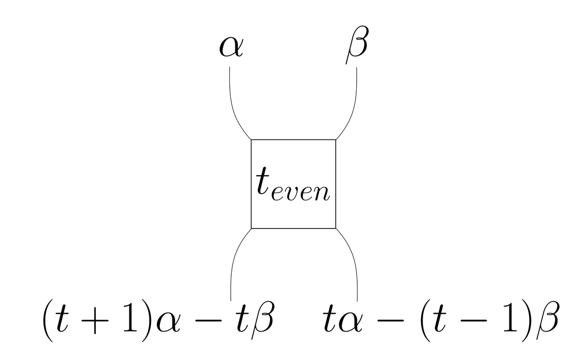
$$= \frac{t_{n+1} a_n + b_n}{a_n} \; \blacksquare$$

Tangle Labeling

Lemma: The end point labels of an odd component with t twists, the α and β endpoints are the strands that are entering the tangle component, is given by

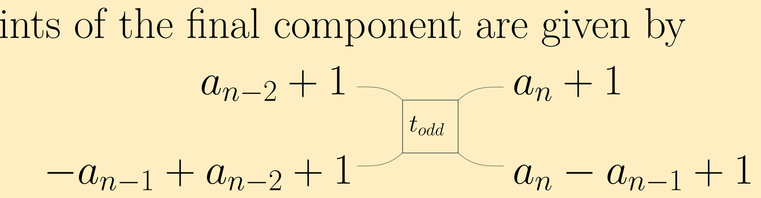


And the end point labels of an even component with t twists, the α and β endpoints are the strands entering the component, is given by

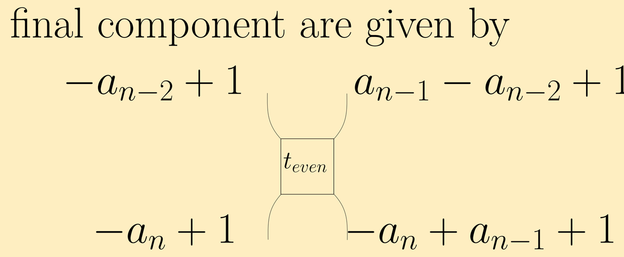


Theorem: $det(C_T) = a_n b_n$

Claim: If a tangle has an odd number of components, the endpoints of the final component are given by

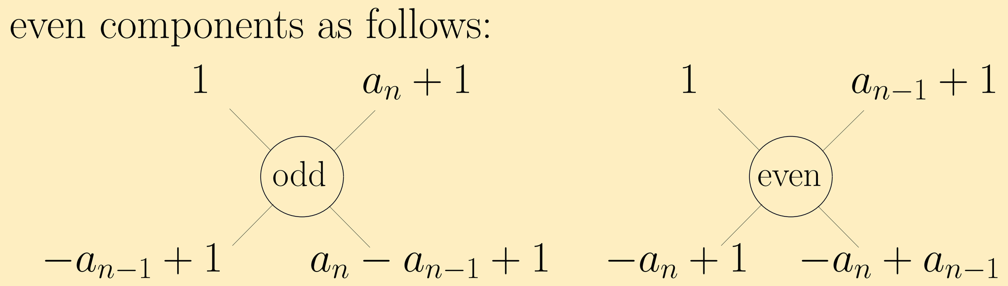


And if a tangle has an even number of components, the endpoints of the final component are given by



This claim can be proven using induction.

From this claim we can construct tangles with both odd and even components as follows:



The determinant of the coloring matrix of the odd component tangle is

$$|1(a_n - a_{n-1} + 1) - (a_n + 1)(a_{n-1} + 1)| =$$

$$|a_n - a_{n-1} + 1 + a_n a_{n-1} - a_n + a_{n-1} - 1| =$$

$$|a_n a_{n-1}|$$

and the determinant of the coloring matrix of the even component tangle is

$$|1(-a_n + a_{n-1} + 1) - (a_{n-1} + 1)(-a_n + 1)| =$$

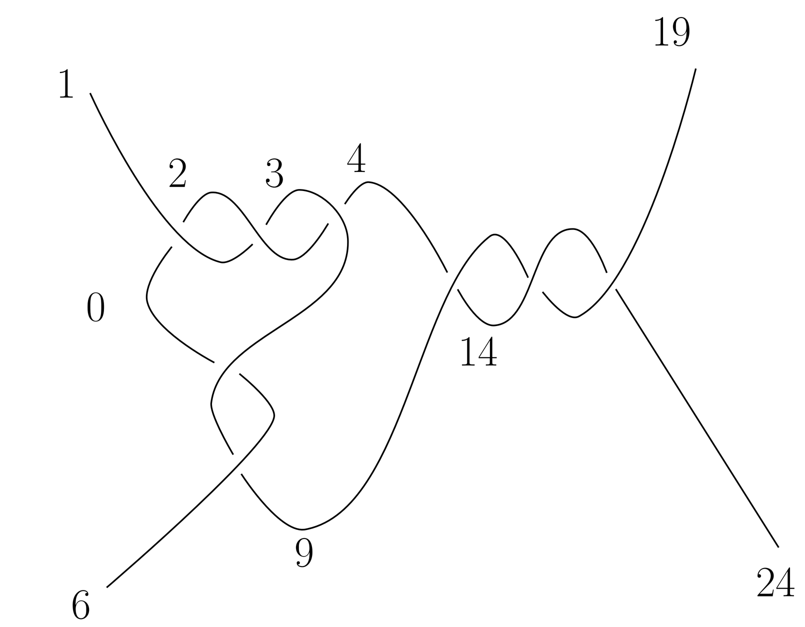
$$|-a_n + a_{n-1} + 1 + a_n a_{n-1} + a_n - a_{n-1} - 1| =$$

$$|a_n a_{n-1}|$$

Since $a_{n-1} = b_n$, $det(C_T) = a_n b_n$.

Example of Theorem

As an example of our theorem we will be examining the (3,-2,-3) tangle.



We begin by finding the determinant of the tangle by first labeling the strands, as shown in the graphic. We see that the determinant of the tangle is

$$\det \begin{bmatrix} 1 & 19 \\ 6 & 24 \end{bmatrix} = -90$$

Thus the tangle has a determinant of 90.

Next we will compute the rational number of the tangle.

$$t_n = -3 + \cfrac{1}{-2 + \frac{1}{3}} = -\frac{18}{5} = \frac{N_t}{D_t}$$

It is easy to see that $N_t * D_t = -18 * 5 = 90$, which is what we found was the determinant of the tangle.

Thus the tangle is 3 and 2-colorable with the numerator closure and 5-colorable when the tangle has the denominator closure.

Future Directions

- Investigate the colorability of closures and what must be done to close the tangle to maintain the coloring.
- Consider what may happen to the colorability when you have a 3 string tangle and eventually generalize it to a n string tangle.
- Explore how local knotting may effect our tangle fraction.

References and Acknowledgements

- Office of Research and Sponsored Programs, UWEC.
- Department of Mathematics, UWEC.
- This study was supported by the UW-Eau Claire Blugold Fellowship Program which is jointly funded by Differential Tuition and The Foundation
- Graphics computed and rendered using *Inkscape*, *Maple* 14 and LaTeX.
- Template created by Philippe Dreuw and Thomas Deselaers of Jacobs University: Computational Physics and Biophysics Group, 2007.

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