**Analysis of Stability Regions of Numeric Methods Using the Time Scale Calculus**

Erin Ferrell and Adam Gordon
Faculty Mentor: Dr. Chris Ahrendt
University of Wisconsin-Eau Claire

**1. The Time Scale Calculus**

**Definition.** A time scale, denoted \( \mathbb{T} \), is a nonempty, closed subset of \( \mathbb{R} \). Let \( \mathbb{T} \). Cantor sets \( \mathbb{T} = \{ 1, 2, 3 \} \) at \( \mathbb{T} = \{ 1, 3, 5, 10, 11 \} \).

**Examples.** A time scale can be \( \mathbb{Z} \), \( \mathbb{R} \), Cantor sets \( \mathbb{T} = \{ 1, 2, 3 \} \) at \( \mathbb{T} = \{ 1, 3, 5, 10, 11 \} \).

**Definition.** If \( \mathbb{T} \) is a time scale and \( t \in \mathbb{T} \), then the forward jump operator \( \sigma \) : \( \mathbb{T} \to \mathbb{T} \) and the backward jump operator \( \rho \) : \( \mathbb{T} \to \mathbb{T} \) are defined to be

\[
\sigma(t) = \inf \{ s \in \mathbb{T} : s > t \} \quad \text{and} \quad \rho(t) = \inf \{ s \in \mathbb{T} : s < t \},
\]

respectively.

**Definition.** Let \( \mu : \mathbb{T} \to [0, \infty) \) be defined by \( \mu(t) = \sigma(t) - t \). This is the graininess function.

**Definition.** An isolated time scale is a time scale where \( \sigma(t) = t \) and \( \rho(t) < t \) for all \( t \in \mathbb{T} \). One important attribute of an isolated time scale is that it can be ordered so that \( \mathbb{T} = \{ t_0, t_1, t_2, \ldots \} \) where \( t_0 < t_1 < t_2 < \ldots \).

**Definition.** The forward difference operator on a function \( f : \mathbb{T} \to \mathbb{R} \) is defined by \( \Delta f(t) = f(t+1) - f(t) \).

**Notation.** Since \( \mathbb{T} \) is isolated, we use the notation \( f(\rho(t)) = f_h(t) = f_{\Delta} \) is the corresponding Butcher tableau. Given the tableau,

\[
\begin{array}{c|c}
\beta & c_0 \\
\hline
b_0 & a_0
\end{array}
\]

the corresponding Runge-Kutta equation is \( y_{n+1} = y_n + h f_{\Delta}(t_n) + \beta k_1 + \beta k_2 + \beta k_3 + \beta k_4 \), where \( k_1 = f(t_n, y_n, t_n, h), k_2 = f(t_n + \frac{h}{2}, y_1, t_n + \frac{h}{2}, h), k_3 = f(t_n + h, y_4, t_n + h, h) \). The tableau that represents the Runge-Kutta method we are using is

\[
\begin{array}{c|c}
0 & 0 \\
\hline
\frac{1}{2} & \frac{1}{2}
\end{array}
\]

**3. The Runge-Kutta Method**

Now we will apply this analysis of the stability region to the well-known Runge-Kutta method. For simplicity we will look at an explicit Runge-Kutta method of order 2, which has the following equation \( y_{n+1} = y_n + h f_{\Delta}(t_n) \). One of the ways that Runge-Kutta methods are organized is by a Butcher tableau. Given the tableau,

\[
\begin{array}{c|c}
\beta & c_0 \\
\hline
b_0 & a_0
\end{array}
\]

Using the equation for the explicit Runge-Kutta method, the test equation and consider \( \mathbb{T} = \mathbb{C} \), we have \( y_0 = y_0 \). We can write the Euler equation as \( y_1 = \frac{y_0}{h} f_{\Delta}(t_0) \). From this we can obtain the dynamic equation \( y_1 = \frac{y_0}{h} f_{\Delta}(t_0) \). Now, using properties of the given exponential we can determine the region where \( e_{\Delta}(t) \) converges to 0. Note that \( e_{\Delta}(t) \) is for all \( t \) in the domain of the solution. Therefore, we have

\[
e_{\Delta}(t) = \frac{y_0}{h} f_{\Delta}(t_0) \to (1 + \frac{h}{2} \beta) \frac{y_0}{h} f_{\Delta}(t_0) = (1 + \frac{h}{2} \beta) \frac{y_0}{h} f_{\Delta}(t_0),
\]

which converges 0 when \( \lim_{n \to \infty} (1 + \frac{h}{2} \beta) \to 0 = 0.0 \) Hence, when \( h \leq \frac{1}{2} \beta \), \( e_{\Delta}(t) \) to 0 on the time scale \( \mathbb{T} = \mathbb{C} \) in the same region as above.

**References**


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