Introduction

A link is a collection of strings tangled together with their ends fused together. Given any link there is a two-dimensional surface it bounds. The genus of this surface gives a measure of the complexity of the link. In this project, we study the analogous measure of complexity given by a generalization of a surface called a C-complex. In order to show that this measure captures some information, we present an infinite family of links for which this new measure of complexity is arbitrarily high. Pictured below are examples of a bounded surface and a C-complex.

Questions and Summary

For a link L the complexity of L, C(L), is defined as the minimum of the complexity of all C-complexes bounded by L. The complexity of a C-complex is defined to be the number of curves needed to “fill up” the complex. Given a C-complex and the curves on it, one can write a matrix by counting linking number of push-offs of curves.

Examples with high complexity

For Figure 5, C(L) ≤ 2k + 1 because there is a C-complex of complexity exactly 2k + 1.

The signature is the summation of the signs of eigenvalues of the link.

Theorem 1 The absolute value of the signature must be less than or equal to the complexity.

Tools: The linking matrix and signature

Given a C-complex, X, of complexity k. There are k curves which fill up X, l_1, l_2, ..., l_k. There is a k × k matrix A^L_{ij} whose \( a_{ij} \) entry is \( \text{link}(l_i, l_j) \), where \( l_i \) is the c-pushoff. Linking counts crossings between \( l_i \) and \( l_j \).

For example for Figure 3, the Cimasoni-Florens linking matrix is given by:

\[
H(\omega_1, \omega_2) = \sum (1 - \omega_1^2)(1 - \omega_2^2) A^L_{ij} = (1 - \omega_1)(1 - \omega_2) (\omega_1 + \omega_2)
\]

The signature of L is defined by the summation of the signs of the eigenvalues of the link. The link L is Hermitian, therefore it has real eigenvalues.

For the link of Figure 3 when \( \theta_1 \) and \( \theta_2 \) are very close to \( -1 \), the signature is

\[
\sigma_\gamma(\omega_1, \omega_2) = 1
\]

Since the signature is 1, the complexity of Figure 3 is exactly 1 because \( 1 \leq C(L) \leq 1 \).

Facts:
1. Any complex for L produces the same \( \sigma \).
2. Since a \( k \times k \) matrix has \( k \) eigenvalues, thus \( |\sigma| \leq k \). Thus the absolute value of the signature must be less than or equal to the complexity. In fact this is the proof of Theorem 1.

Examples with high complexity

For Figure 5, C(L) ≤ 2k + 1 because there is a C-complex of complexity exactly 2k + 1.

By solving this recurrence relation, we get that \( G_1 \) has eigenvalues (up to sign) \( p_1 = -\sin\left(k(1)(\pi - \frac{2\theta_1}{\pi})\right) \) where \( \omega_1 = e^{i\theta_1} \) and \( \omega_2 = e^{i\theta_2} \), and \( G_2 \) has eigenvalues (again, up to sign) \( q_i = -\sin\left(k(1)(\pi - \frac{2\theta_2}{\pi})\right) \).

For \( \theta_1 + \theta_2 \) and \( \theta_1 - \theta_2 \) positive and very close to 0, \( p_1, \ldots, p_k \) and \( q_1, \ldots, q_k \) are all negative. Unfortunately the last eigenvalue is positive. Thus, for this choice of \( \theta_1 \) and \( \theta_2 \), \( \sigma_\gamma(\omega_1, \omega_2) = -2k + 1 \).

Future Research

• If we add n-additional positive twists, we hope to be able to compute the complexity more precisely.

References


Advisor: Dr. Christopher Davis

We gratefully acknowledge support by the Blugold Commitment Differential Tuition Grants Program and the Trends in Undergraduate Math Research Symposium.