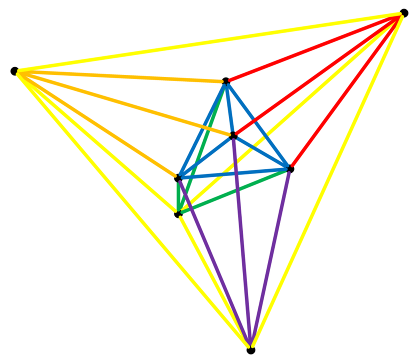

n-Dimensional Semi-Hypercubes and the Algebras Associated with their Hasse Graphs

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The Problem

There is a Hasse graph associated with every n-dimensional polytope. Each level of the graph represents the number of k-dimensional faces that remain fixed under a given automorphism (or symmetry) of the polytope. For each automorphism, we determine a polynomial, such as $f(t) = 4 - 2t - 2t^4 + t^5$, where the power of t represents the length of each path. The coefficient of t^0 is the number of points, the coefficient of t^1 is the number of paths 1 level apart, ..., the coefficient of t^{n+1} is the number of paths $n+1$ levels apart.

Our goal is to be able to predict how automorphisms of the semi-hypercubes act using the Hasse graphs of the fixed k-faces to obtain a generating function for the Hasse graph polynomial. Once we determine the generating function associated with each automorphism, we can determine the structure of the algebra.

Background

To create an n-dimensional semi-hypercube, we first consider a unit n-cube with one vertex at the origin. We then keep only those verticies with an even number of 1's, and form new simplex and semi-hypercube facets.

Definition: *We define $\mathcal{L}_\sigma = \{l_1, l_2, l_3, ..., l_n\}$ where l_i represents the number of times a cycle of length i is used in the permutation σ, a permutation that permutes the coordinates of the vertices in the semi-hypercube.*

Definition: *We define $q = \{q_1, q_2, q_3, ..., q_n\}$ where $q_i \leq l_i$. $q \vdash a$ means $\sum_{i=1}^{a} iq_i = a$. We define p similarly.*

Definition: *We define σ to be evenly composed (EC) if σ contains only even length permutations. Otherwise it is oddly composed (OC).*

Number of Fixed Faces when Coordinate Values are Permuted

	Simplicies		Semihypercubes	
k (dim)	OC σ	EC σ	OC σ	EC σ
0	$2^{\sum l_i - 1}$	$2^{\sum l_i}$	0	0
1	0	0	$\sum_{p \vdash 2} 2^{\sum l_i - \sum p_i} \prod \binom{l_i}{p_i}$	
2	$\sum_{q \vdash 3} 2^{\sum l_i - 1} \prod \binom{l_i}{q_i}$	0	0	0
≥ 3	$\sum_{q \vdash k+1} 2^{\sum l_i - 1} \prod \binom{l_i}{q_i}$	0	$\sum_{p \vdash k} 2^{\sum l_i - \sum p_i} \prod \binom{l_i}{p_i}$	

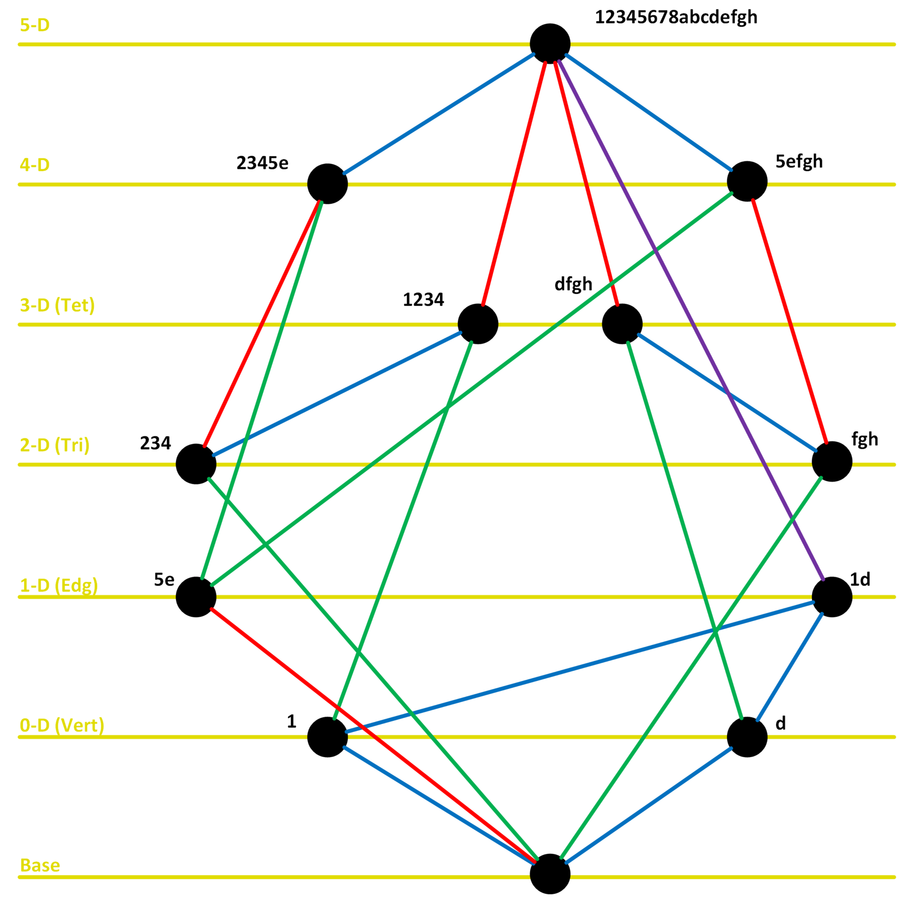
Method

Once we obtained the formulas for the number of fixed k-faces, we determined the number of fixed k-faces contained within a given fixed face of the semi-hypercube. We did this by treating the p or q for the fixed face as a new $\mathcal{L}$, and reapplying the formulas. If we apply this method for each permutation, we can count the number of paths of each length throughout the Hasse Graph.

We created two distinct computer programs to compute the generating function. One program created the Hasse graph, like on the right, and counted the paths the long-hand way, while the other used the functions given below to obtain the generating function.

Once we had the generating functions for the n-dimensional semi-hypercube under permutations of the coordinates of the vertices, we started to look at how the remaining automorphisms of the semi-hypercube will affect which faces are fixed. These come from adding an n-tuple of $\mathbb{Z}_2$ to the coordinate values of every vertex, where a 1 changes the value and a 0 doesn't. Because of the way that we defined our semi-hypercube, we can only change an even number of coordinate values, so there can only be an even amount of 1's in the n-tuple.

$\mathcal{L} = (0, 1, 1, 0, 0) \Rightarrow (1)(243)(5e)(6a7c8b)(d)(fhg)$



Results

We found that there are two formulas to encompass every situation: one if $\mathcal{L}$ is evenly composed, and one if $\mathcal{L}$ is oddly composed. Below are only a part of each formula, and the rest of the formulas use the same idea. However, because the number of fixed faces of dimension less than three is not given by the same formula as fixed faces of higher dimension, different formulas are needed to count the paths that pass through, or end on, dimensions less than three.

Evenly Composed

$$[3^+ \to 3^+] \sum_{k=3}^{n-j} \left(\sum_{p \vdash k+j} \left((-1)^{\sum p_i + k + j} \left[2^{\sum l_i - \sum p_i} \prod \binom{l_i}{p_i} \right] \left[\sum_{\substack{p' \vdash k \\ p' \subseteq p}} (-1)^{\sum p'_i + k} 2^{\sum p_i - \sum p'_i} \prod \binom{p_i}{p'_i} \right] \right) \right) \text{ when } j \geq 0$$

Oddly Composed

$$[3^+ \to 3^+] \sum_{k=3}^{n-j} \left(\sum_{q \vdash k+j+1} \left((-1)^{\sum q_i + k + j + 1} \left[2^{\sum l_i - 1} \prod \binom{l_i}{q_i} \right] \left[\sum_{\substack{q' \vdash k+1 \\ q' \subseteq q}} (-1)^{\sum q'_i + k + j + 1} \prod \binom{q_i}{q'_i} \right] \right) \right.$$

$$\left. + \sum_{p \vdash k+j} \left((-1)^{\sum p_i + k + j} \left[2^{\sum l_i - \sum p_i} \prod \binom{l_i}{p_i} \right] \left[\sum_{\substack{q' \vdash k+1 \\ q' \subseteq p}} (-1)^{\sum q'_i + k + 1} 2^{\sum p_i - 1} \prod \binom{p_i}{q'_i} + \sum_{\substack{p' \vdash k \\ p' \subseteq p}} (-1)^{\sum p'_i + k} 2^{\sum p_i - \sum p'_i} \prod \binom{p_i}{p'_i} \right] \right) \right) \text{ when } j \geq 0$$

After getting the above formulas, we ran some preliminary tests on the computer program that generates the Hasse graph to look at the implication of the n-tuples. We found that when an odd number of elements in a specific cycle of σ are changed, the formulas are ruined in a yet undetermined way. For example, $\mathcal{L} = (1, 1, 1, 0, 0, 0) \Rightarrow \sigma = (123)(45)(6)$. Adding $(1, 0, 0, 0, 1, 0)$ to each coordinate value would then change the first and the fifth coordinate value of each vertex. This would ruin the formula because in the cycle (123), only one value (odd), the first coordinate value, is changed. However, adding $(1, 1, 0, 1, 1, 0)$ would not ruin the formula, because in the cycle (123), two values (even), the first and second, are changed, and in the cycle (45) two values (even), the forth and fifth, are also changed.

References

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Future Direction

We will explore what happens when we add an n-tuple of $\mathbb{Z}_2$ to each coordinate after they have been permuted. This will complete the full automorphism group, and we will be able to understand the algebra associated with the Hasse graphs of n-dimensional semi-hypercubes. We will then extend our research to the remainder of the Coxeter Groups, beginning with the icosahedron.

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