

More Congruences for the k -regular Partition Function

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Abstract

A partition of a number n is a list of positive integers whose sum is n . For example, $4 + 2 + 1$ and $4 + 1 + 1 + 1$ are both partitions of 7. It can be shown that 4 has 5 partitions, 9 has 30 partitions, 14 has 135 partitions, and Srinivasa Ramanujan proved the following beautiful result: the number of partitions of $5n + 4$ is divisible by 5 for any nonnegative integer n . The k -regular partition function counts the number of partitions of n whose parts are not divisible by k . In 2012, for particular values of k , David Furcy and David Penniston found many families of integers whose number of k -regular partitions is divisible by 3. In this paper, I extend their results to larger values of k and provide an overview of the methodology used to arrive at the result. In the interest of brevity, only a sketch of the proof is given.

Roadmap

This project lies at the intersection of mathematics and computation, so it necessitates the presentation of a significant amount of background material. The first three sections introduce partitions, explain congruences, and demonstrate how to calculate both. Though the reader may be unfamiliar with these topics, the introductory material largely deals with relatively simple concepts like integers and divisibility. Several of the new concepts are endnoted with exercises that can be completed with paper and a pencil. The exercises are not designed to be difficult, but they will provide a familiarity with the material that mere reading cannot achieve. The following three sections describe the methodology I used for the project, present my new results, and provide an abridged proof of the results. The penultimate section discusses my results within the context of the field and offers some possibilities for future work. Finally, the last section is dedicated to acknowledging the contributions of those who made this project possible. An answer key for the exercises is provided as an appendix.

Introduction to Partitions

An *integer* is a whole number that can be negative, positive, or zero. Thus, -1 , 2 , and 0 are all integers, while 0.5 and $1/3$ are not. Given an integer n , a *partition* of n is a nonincreasing sequence of positive integers that add up to it. For example, $3 + 1$ is a partition of $n = 4$. The numbers present in a partition are called *parts*. A natural

question arising from this definition is: Given some nonnegative integer n , how many partitions does it have? As an example, figure 1 provides all the partitions of 5.

- 5
- 4 + 1
- 3 + 2
- 3 + 1 + 1
- 2 + 2 + 1
- 2 + 1 + 1 + 1
- 1 + 1 + 1 + 1 + 1

Figure 1. The partitions of $n = 5$.

We find that 5 has a total of 7 partitions (note that due to the nonincreasing condition, we do not consider $1 + 4$ or $1 + 3 + 1$ to be partitions). For convenience, we define a function $p(n)$ which, given an integer n , returns the number of partitions of n . Then, as we just found, $p(5) = 7$.¹ Table 1 features the first eleven values of $p(n)$. Note that $p(0)$ is equal to 1 because there exists only one list of positive integers whose sum is 0, namely the empty list. Since the sum of any list of positive integers is at least 0, we have that $p(x) = 0$ for all negative integers x .

Table 1. Some values of $p(n)$.

n	$p(n)$
0	1
1	1
2	2
3	3
4	5
5	7
6	11
7	15
8	22
9	30
10	42

It is possible to create new functions similar to $p(n)$ by placing restrictions on the partitions. Given a positive integer k , a partition is called k -regular if none of its parts are divisible by k . The 2-regular partitions of 5 are listed in figure 2.

- 5
- 3 + 1 + 1
- 1 + 1 + 1 + 1 + 1

Figure 2. The 2-regular partitions of $n = 5$.

The remaining partitions of 5 are not 2-regular because they contain parts, namely 4 and 2, that are divisible by 2:

- 4 + 1
- 3 + 2
- 2 + 2 + 1
- 2 + 1 + 1 + 1

Figure 3. The non-2-regular partitions of $n = 5$.

Similar to $p(n)$, we define the k -regular partition function $b_k(n)$ to return the number of k -regular partitions of n . Figure 2 shows that $b_2(5) = 3$.² The k -regular partition function for certain values of k is the focus of this paper.

Congruences

A *congruence* is a statement about divisibility, and there exist many congruences for the partition function. The first five values of $p(5n + 4)$ are in table 2. Note that each value ends in either 0 or 5, meaning that each is divisible by 5.

Table 2. Some values of $p(5n + 4)$.

n	$5n + 4$	$p(5n + 4)$	$p(5n + 4)$, factored
0	4	5	$5 \cdot 1$
1	9	30	$5 \cdot 6$
2	14	135	$5 \cdot 27$
3	19	490	$5 \cdot 98$
4	24	1575	$5 \cdot 315$

It turns out that this phenomenon continues indefinitely; that is, every number in the infinite list $p(29), p(34), p(39), \dots$ is also divisible by 5. This pattern is part of a set of results known as *Ramanujan’s congruences*, which were among the first congruence results to be proven about the partition function.

$p(5n + 4)$ is divisible by 5.

$p(7n + 5)$ is divisible by 7.

$p(11n + 6)$ is divisible by 11.

Figure 4. Ramanujan’s congruences, for any integer n .

Srinivasa Ramanujan proved the first two congruences of figure 4 and stated the third without proof, and after his death, G. H. Hardy used material from Ramanujan’s notebooks to reconstruct a proof of the third result (Ramanujan 2000). Ramanujan’s work still inspires new research today, and there is even a journal dedicated to results that stem from his discoveries and creations.

Note that 5, 7, and 11 are all prime numbers, i.e., each is divisible only by 1 and itself. The curious reader may wonder whether congruences of the form “ $p(An + B)$ is divisible by A ” exist for other primes A , but it turns out that 5, 7, and 11 are unique in this regard (Ahlgren and Boylan 2003). On a brighter note, for any positive integer M that is not divisible by 2 or 3, there always exist congruences of the form “ $p(An + B)$ is divisible by M ” (Ahlgren and Ono 2001). Johansson (2012) gives one such congruence for $M = 13$:

$$p(711647853449n + 485138482133) \text{ is divisible by } 13.$$

Unfortunately, Ahlgren and Ono’s proof that congruences always occur is not constructive: it does not explicitly show how to find congruences, but merely demonstrates that they exist.

Perhaps unsurprisingly, congruence results akin to those satisfied by the partition function $p(n)$ also occur for the k -regular partition function $b_k(n)$ for various values of k . Consider $k = 13$. For n between 0 and 12, $b_{13}(n)$ is the same as $p(n)$, since no partition for these numbers can contain parts divisible by 13. However, for larger n , $b_{13}(n)$ excludes some of the available partitions, and so it grows more slowly than $p(n)$.

Table 3. $p(n)$ versus $b_{13}(n)$.

n	$p(n)$	$b_{13}(n)$
0	1	1
10	42	42
20	627	612
30	5604	5302
40	37338	34193
50	204226	181014

Similar to Ramanujan with $p(n)$, there is a pattern in the 13-regular partition function:

Table 4. Some values of $b_{13}(9n + 7)$.

n	$9n + 7$	$b_{13}(9n + 7)$	$b_{13}(9n + 7)$, factored
0	7	15	$3 \cdot 5$
1	16	228	$3 \cdot 76$
2	25	1881	$3 \cdot 627$
3	34	11496	$3 \cdot 3832$
4	43	57360	$3 \cdot 19120$

Calkin et al. (2008) proved that the congruence

$$b_{13}(9n + 7) \text{ is divisible by } 3$$

holds true for any nonnegative integer n .

Moreover, Webb (2011) proved a *recurrence*—a relationship between a function and its previous values—for the 13-regular partition function. The values of $b_{13}(n)$ in table 5 demonstrate the recurrence:

Table 5. Recurrence values for $b_{13}(n)$.

n	$b_{13}(3n + 1)$	$b_{13}(9n + 4)$	$b_{13}(3n + 1) + b_{13}(9n + 4)$	$b_{13}(3n + 1) + b_{13}(9n + 4)$, factored
0	1	5	6	$3 \cdot 2$
1	5	100	105	$3 \cdot 35$
2	15	972	987	$3 \cdot 329$
3	42	6450	6492	$3 \cdot 2164$
4	100	34193	34293	$3 \cdot 11431$
5	228	154293	154521	$3 \cdot 51507$

The recurrence can therefore be stated as

$$b_{13}(3n + 1) + b_{13}(9n + 4) \text{ is divisible by } 3,$$

and it holds for all nonnegative n .

Suppose we replace the n in the recurrence by $3n + 2$. This is equivalent to only considering $n = 2$, then $n = 5$, then $n = 8$, and so on. When we make the replacement, the recurrence becomes

$$b_{13}(3(3n + 2) + 1) + b_{13}(9(3n + 2) + 4) \text{ is divisible by } 3,$$

which simplifies to

$$b_{13}(9n + 7) + b_{13}(27n + 22) \text{ is divisible by } 3.$$

Since we simply selected particular values from the original recurrence, this new statement is also true. Observe that the expression from the congruence is present in this new recurrence. Since $b_{13}(9n + 7)$ is divisible by 3 and $b_{13}(9n + 7) + b_{13}(27n + 22)$ is also divisible by 3, it must be the case that

$$b_{13}(27n + 22) \text{ is divisible by } 3.$$

In other words, the recurrence generated a new congruence out of the old one. Even better, this process can be repeated indefinitely, with each new congruence generating yet another. The first few congruences Webb generated for the 13-regular partition function are located in table 6. To avoid repetition, I elide the “is divisible by 3” portions of the congruences.³

Table 6. Some of Webb’s congruences for $b_{13}(n)$ divisible by 3.

Number of Uses of the Recurrence	Congruence
0	$b_{13}(9n + 7)$
1	$b_{13}(27n + 22)$
2	$b_{13}(81n + 67)$
3	$b_{13}(243n + 202)$
4	$b_{13}(729n + 607)$

All of the (infinitely many) congruences can be written as a single statement: for all nonnegative integers m and n ,

$$b_{13}\left(3^{m+2}n + \frac{5 \cdot 3^{m+1} - 1}{2}\right) \text{ is divisible by } 3.^4$$

My congruences for the k -regular partition function are similar in form to this result, though the proof is slightly different.

Congruence results have been found for many other values of k (Ahlgren and Lovejoy 2001; Andrews, Hirschorn, and Sellers 2010; Calkin et al. 2008; Dandurand and Penniston 2009; Furcy and Penniston 2012; Gordon and Ono 1997; Lovejoy 2001; Penniston 2008). Furcy and Penniston deal with values of k that are less than 50. I extend their results by proving congruences for $b_k(n)$ for particular values of k between 50 and 100.

Calculating Partition Functions

The k -regular partition numbers can be constructed out of the values of $p(n)$. Therefore, in order to search for congruences involving $b_k(n)$, it is helpful to know the values of $p(n)$. However, calculating them by hand is time consuming because $p(n)$ grows quickly; for example, $p(30)$ is 5604 and $p(100)$ is 190,569,292. Luckily, a formula arising out of the pentagonal number theorem proved by Leonhard Euler

makes it possible to calculate $p(n)$ without building any partitions (Enestrom 1910). The n^{th} pentagonal number is defined to be $G(n) = (3n^2 - n) / 2$. As the name implies, this formula has a geometric interpretation: $G(n)$ is equal to the number of dots appearing in the n^{th} set of nested pentagons like those in figure 5.⁵

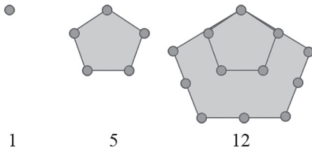


Figure 5. Count the dots: calculating $G(1)$, $G(2)$, and $G(3)$.

The *generalized pentagonal numbers* result from placing all integers in the formula.

Table 7. Some generalized pentagonal numbers.

n	$G(n)$
0	0
1	1
-1	2
2	5
-2	7
3	12
-3	15
4	22
-4	26
5	35
-5	40

Recall that a recurrence is a relation between a function and its previous values. Euler proved the following striking recurrence for $p(n)$:

$$p(n) = p(n - 1) + p(n - 2) - p(n - 5) - p(n - 7) + p(n - 12) + p(n - 15) - - + + \dots,$$

where the alternating pattern of adding twice and subtracting twice continues indefinitely, and the numbers that are subtracted from n are the generalized pentagonal numbers. Also recall that $p(x) = 0$ when x is negative, so the sum terminates once it reaches a generalized pentagonal number larger than n . Using the recurrence, if the values of $p(0)$, $p(1)$, \dots , $p(n - 2)$, and $p(n - 1)$ are all known, then we can calculate $p(n)$ without much extra work. In fact, only a small subset of these partition numbers needs to be known. For example, we can use the values in table 1 to calculate $p(11)$:

$$\begin{aligned}
 p(n) &= p(n - 1) + p(n - 2) - p(n - 5) - p(n - 7) + p(n - 12) + p(n - 15) - - + + \dots \\
 p(11) &= p(11 - 1) + p(11 - 2) - p(11 - 5) - p(11 - 7) + p(11 - 12) + p(11 - 15) - - + + \dots \\
 p(11) &= p(10) + p(9) - p(6) - p(4) + p(-1) + p(-4) - - + + \dots \\
 p(11) &= p(10) + p(9) - p(6) - p(4) \\
 p(11) &= 42 + 30 - 11 - 5 \\
 p(11) &= 56
 \end{aligned}$$

We find that to calculate $p(11)$, we only need to know $p(10)$, $p(9)$, $p(6)$, and $p(4)$.

Using the recurrence is much less tedious than enumerating all partitions by hand, and it can be easily implemented on a computer to calculate all the partition numbers between 0 and n for reasonably large n .⁶ Alternatively, if we are interested in a single value $p(n)$ rather than all of the values $p(0), p(1), \dots, p(n-1), p(n)$, there exist efficient algorithms to calculate it without needing any of the previous values (Johansson 2012). From this point forward, all of the partition values I will mention were calculated using Euler's recurrence in a computer program.

Methodology

The method I used to prove my congruence results is due to Furcy and Penniston. It can be broken down into three steps: calculate partition values, search for congruences, and prove congruences. The nature of the proof restricts its applicability to values of k such that 3 divides $k-1$. Hereafter, all values of k mentioned will have this property unless stated otherwise.

I began the calculate step by implementing Euler's recurrence in the C programming language and using this program to generate all values of $p(n)$ for n up to two billion. The partition function grows quickly— $p(10^9)$ has 35,219 digits—so it was not feasible to store its true values. Luckily, I was only concerned with divisibility by 3, so every partition number could be stored as either 0, 1, or 2, depending on whether $p(n)$, $p(n)-1$, or $p(n)-2$ was divisible by 3, respectively. This is called considering $p(n)$ modulo 3. Thus, every partition number I calculated consisted of a single digit. Even so, the set of all partition numbers took up 500 megabytes of space, approximately 75% of the capacity of a standard CD.

The computation was performed *serially*—one partition number at a time—and took approximately eight days to complete. Once it was done, for each k between 50 and 100, I used the values to calculate $b_k(n)$ for all n less than 10,000,000. These sets of ten million values were the *search spaces* of my project. I decided the size of the spaces by trial and error; this size eliminated all false positives from the search step but was small enough to calculate in a reasonable amount of time (a few seconds). Furcy and Penniston searched for congruences of the form

$$b_k(3^A n + B) \text{ is divisible by } 3,$$

where k is less than 50, A is between 1 and 10 and B is between 0 and $3A-1$. I searched for congruences of this same form, but with k between 50 and 100. The search step was a simple brute-force search: it looked at every possibility of A and B and checked if $b_k(3^A n + B)$ is divisible by 3 for all n that kept $3^A n + B$ within my search space. If the divisibility held for all n that I checked, then I added a congruence to the list of candidates. Next, again using a brute-force search, I looked for recurrences of the form

$$b_k(3^A n + B) + b_k(3^C n + D) \text{ is divisible by } 3,$$

and

$$b_k(3^A n + B) - b_k(3^C n + D) \text{ is divisible by } 3,$$

where A is less than C and both are less than or equal to 10. As we saw earlier, such a recurrence could potentially be used to generate new congruences out of old ones. Indeed, analogous to Webb's result for $b_{13}(n)$, I found recurrences for certain $b_k(n)$ that generated infinitely many new congruences.

The final step was to prove the congruences and recurrences found in the search step. The problem is that a congruence like

$$b_{67}(729n + 301) \text{ is divisible by } 3$$

is a statement about infinitely many values of n , while the search step of course checks only finitely many. While I will not go into detail here, it is possible to construct a mathematical object known as a *modular form* that encodes the values of $b_k(n)$. Additionally, there is a transformation that can be applied to a form called the *Hecke operator*, which can yield a function that might correspond to a candidate congruence or recurrence found in the search step. Once a candidate congruence is associated with a modular form, a theorem of Jacob Sturm tells us that if the congruence holds up to a certain point, then it must hold forever (Sturm 1987).

The proof step therefore consisted of three parts. First, I constructed modular forms that corresponded to each of the congruences and recurrences. Then, depending on the Sturm bound attached to each form, I calculated the necessary values of the k -regular partition function using the values of $p(n)$ and verified that each congruence and recurrence held true. Finally, I applied the relevant recurrences to each congruence to generate families of numbers whose k -regular partition numbers are divisible by 3.

Results

Theorem 1. Let n and m be any pair of nonnegative integers. Then, all of the following are divisible by 3:

$b_{67}\left(3^{2m+6}n + \frac{5 \cdot 3^{2m+5} - 11}{4}\right)$	$b_{67}\left(3^{2m+7}n + \frac{11 \cdot 3^{2m+6} - 11}{4}\right)$
$b_{70}\left(3^{2m+6}n + \frac{13 \cdot 3^{2m+5} - 23}{8}\right)$	$b_{70}\left(3^{2m+7}n + \frac{23 \cdot 3^{2m+6} - 23}{8}\right)$
$b_{79}\left(3^{2m+4}n + \frac{11 \cdot 3^{2m+3} - 13}{4}\right)$	$b_{79}\left(3^{2m+5}n + \frac{5 \cdot 3^{2m+4} - 13}{4}\right)$
$b_{91}\left(3^{2m+4}n + \frac{5 \cdot 3^{2m+3} - 15}{4}\right)$	$b_{91}\left(3^{2m+5}n + \frac{11 \cdot 3^{2m+4} - 15}{4}\right)$

Note that for each pair of families, the first family has even powers of 3 attached to n , while the second has odd powers of 3. Also, the powers of 3 attached to n increase by 2 when m is increased by 1. Thus, each pair of families generates congruences containing all powers of 3 greater than or equal to the even power of 3 calculated when $m = 0$. Table 8 illustrates this by showing the first few congruences from the families for $b_{67}(n)$.

Table 8. Some of my congruence results for $b_{67}(n)$ divisible by 3.

m	Congruence from Family 1	Congruence from Family 2
0	$b_{67}(729n + 301)$	$b_{67}(2187n + 2002)$
1	$b_{67}(6561n + 2731)$	$b_{67}(19683n + 18040)$
2	$b_{67}(59049n + 24601)$	$b_{67}(177147n + 162382)$

As an example of an individual congruence, values of $b_{67}(729n + 301)$ are placed in table 9.

Table 9. Some values of $b_{67}(729n + 301)$.

n	$729n + 301$	$b_{67}(729n + 301)$, factored
0	301	$3 \cdot 3288304305337708$
1	1030	$3 \cdot 24111764832321169867987562139516$
2	1759	$3 \cdot 36477912261775772450764503576351$
3	2488	$3 \cdot 555816587146753951030956700341289285382444975430921$
4	3217	$3 \cdot 16563511516295866880289667762269701587577286136177203182764$

Similar to Furcy and Penniston, I also found examples of congruences with no applicable recurrences in the search space, and vice versa.

Theorem 2. Let n be a nonnegative integer. Then, the following are divisible by 3:

$$\begin{aligned}
 &b_{35}(243n + 180) \\
 &b_{70}(729n + 392) \\
 &b_{70}(2187n + 2093) \\
 &b_{61}(9n + 2) - b_{61}(729n + 362) \\
 &b_{85}(9n + 1) - b_{85}(729n + 361)
 \end{aligned}$$

Recall that my search for congruences was brute-force. This type of search ensures that all congruences that exist within the search space are found, which leads us to my final theorem.

Theorem 3. The congruences generated by Theorem 1 and listed in Theorem 2 account for *all* congruences of the form

$$b_k(3^A n + B) \text{ is divisible by } 3,$$

where the variables $k, n, A,$ and B are integers subject to the following conditions:

- $50 < k \leq 100$
- 3 divides $k - 1$
- n is nonnegative
- $1 \leq A \leq 10$
- $0 \leq B \leq 3^A.$

Sketch of the Proof

Since the proofs of the cases are nearly identical, I will only sketch the proof of the first congruence family in Theorem 1. A candidate congruence found in the search step was

$b_{67}(729n + 301)$ is divisible by 3.

Additionally, I found the recurrence

$$b_{67}(243n + 58) - b_{67}(2187n + 544) \text{ is divisible by } 3.$$

I now show that it is possible to construct modular forms that encode these values of $b_{67}(n)$.

As is customary, I will denote the complex vector space of modular forms of weight k , level N , and Dirichlet character χ as $M_k(\Gamma_0(N), \chi)$ and will write

$$f \in M_k(\Gamma_0(N), \chi)$$

to denote that f is a function in this space. The values k and N are used to calculate the Sturm bound for all modular forms in the space. My modular forms are built out of products of Dedekind's eta function, which is defined as:

$$\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),$$

where $q(z) = e^{2\pi iz}$. There exist criteria that determine whether or not a particular product or quotient involving the eta function is in fact a modular form (Ono 2004).

My first form is

$$g(z) = \eta(67z)\eta(z)^{10205} \in M_{5103}(\Gamma_0(67), \chi_{67}).$$

The definition of the eta function and some elementary techniques in number theory show that $g(z)$ can be expressed as:

$$g(z) = \left(\sum_{n=0}^{\infty} b_{67}(n)q^{n+428} \right) \cdot F_0(z).$$

The sum in parenthesis is called a q -expansion. The function $F_0(z)$ has a q -expansion that is not divisible by 3. Written another way,

$$g(z) = (b_{67}(0)q^{428} + b_{67}(1)q^{429} + b_{67}(2)q^{430} + b_{67}(3)q^{431} + \dots) \cdot F_0(z).$$

We can apply the index 3 Hecke operator (written as T_3) to $g(z)$ and then consider the result's divisibility (or non-divisibility) by 3. Table 10 contains the first six applications of T_3 to $g(z)$, modulo 3. As with $F_0(z)$, each $F_i(z)$ has a q -expansion that is not divisible by 3.

Table 10. Six applications of T_3 to $g(z)$.

i	$g(z)$ after i applications of T_3 , modulo 3
0	$\left(\sum_{n=0}^{\infty} b_{67}(n)q^{n+428}\right) \cdot F_0(z)$
1	$\left(\sum_{n=0}^{\infty} b_{67}(3n+1)q^{n+143}\right) \cdot F_1(z)$
2	$\left(\sum_{n=0}^{\infty} b_{67}(9n+4)q^{n+48}\right) \cdot F_2(z)$
3	$\left(\sum_{n=0}^{\infty} b_{67}(27n+4)q^{n+16}\right) \cdot F_3(z)$
4	$\left(\sum_{n=0}^{\infty} b_{67}(81n+58)q^{n+6}\right) \cdot F_4(z)$
5	$\left(\sum_{n=0}^{\infty} b_{67}(243n+58)q^{n+2}\right) \cdot F_5(z)$
6	$\left(\sum_{n=0}^{\infty} b_{67}(729n+301)q^{n+1}\right) \cdot F_6(z)$

The properties of the Hecke operator imply that each one of the functions in table 10 is congruent modulo 3 to a member of $M_{5103}(\Gamma_0(67), \chi_{67})$. The sixth application of T_3 to $g(z)$ is used to prove the first congruence for the first result in Theorem 1. The bound given by Sturm for $M_{5103}(\Gamma_0(67), \chi_{67})$ is $n = 28917$. I computationally verified that $b_{67}(729n + 301)$ is divisible by 3 for any nonnegative n less than or equal to this bound. With a minor argument about the properties of F_6 , Sturm’s theorem implies that $b_{67}(729n + 301)$ is divisible by 3 for any nonnegative n .

With the congruence proven, all that remains is to demonstrate that the recurrence is true and to apply it to the congruence. A proof similar to the one above, using the modular forms

$$h_1(z) = \eta(67z)\eta(z)^{10205} \in M_{5103}(\Gamma_0(67), \chi_{67})$$

and

$$h_2(z) = \eta(67z)\eta(z)^{39365} \in M_{19683}(\Gamma_0(67), \chi_{67}),$$

shows that the recurrence

$$b_{67}(243n + 58) - b_{67}(2187n + 544) \text{ is divisible by } 3$$

holds for any nonnegative n . Note that h_1 is the same as g from my proof for the congruence. The function h_1 has weight 5103, while h_2 has weight 19683, so we must “scale” h_1 to weight 19683 to get both functions in the same space (which allows the use of Sturm’s theorem). This scaling can be accomplished by multiplying h_1 by

a certain power of the Eisenstein function E_4 . The properties of E_4 imply that this multiplication does not change h_1 modulo 3.

Finally, taking the recurrence and replacing n by $3n + 1$ yields a new recurrence,

$$b_{67}(729n + 301) - b_{67}(6561n + 2731) \text{ is divisible by } 3,$$

which is applicable to the congruence I just proved. The first new congruence generated is therefore

$$b_{67}(6561n + 2731) \text{ is divisible by } 3.$$

Repeated applications of the recurrence yield the first family of congruences of Theorem 1 in the Results section.

Conclusion and Future Work

Furcy and Penniston found fourteen families of congruences for nine different values of k where k is less than 50. I found only eight families of congruences over four values of k between 50 and 100, which suggests that congruences become less common as k increases. However, it is possible that congruences *do* occur for other values of k , but at higher numbers than those in my search spaces. Thus, this work can be extended by increasing the size of the search space and looking at the values of $b_k(n)$ for $k \leq 100$. Attempting to find congruences for higher values of k while keeping the search space the same size might prove to be fruitful, though it seems that any such results would be sparse. It may also be possible to find a different proof method that removes the requirement that $k - 1$ be divisible by 3.

Recall that Ahlgren and Ono showed that there always exist congruences of the form

$$p(An + B) \text{ is divisible by } M$$

when M is not divisible by 2 or 3. It is not unreasonable to suspect that an analogous result exists for the k -regular partition function. Therefore, a general extension of this work would be to prove that for certain values of k , congruences of the form

$$b_k(An + B) \text{ is divisible by } M$$

always occur for values of M , possibly subject to restrictions as in the $p(n)$ case.

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Notes

1. Write the partitions of $n = 4$. How many did you get (i.e., what is $p(4)$)?
2. Write all the 3-regular partitions of $n = 5$. How many are there (i.e., what is $b_3(5)$)?
3. Notice that each entry after the first in table 6 is calculated by multiplying the previous entry by 3 and then adding 1. For example, $3 \cdot (9n + 7) + 1 = (27n + 21) + 1 = 27n + 22$. Apply this technique to the last entry to get the next congruence.

4. Set $m = 5$ in the “single statement” expression to verify your answer for the previous question.
5. Copy the diagram for the 3rd pentagonal number. On top of it, draw a pentagon with four dots on each side. Count the dots to get the 4th pentagonal number.
6. Suppose we only knew the values of $p(0)$, $p(1)$, $p(2)$, $p(3)$, and $p(4)$. Use the recurrence and table 1 to find $p(5)$.

Bibliography

- Ahlgren, Scott, and Jeremy Lovejoy. “The Arithmetic of Partitions into Distinct Parts.” *Mathematika* 48, no. 1–2 (2001): 203–11.
- Ahlgren, Scott, and Ken Ono. “Congruence Properties for the Partition Function.” *Proceedings of the National Academy of Sciences of the United States of America* 98, no. 23 (2001): 12,282–84.
- Ahlgren, Scott, and Matthew Boylan. “Arithmetic Properties of the Partition Function.” *Inventiones Mathematicae* 153, no. 3 (2003): 487–502.
- Andrews, G. E., M. D. Hirschorn, and J. A. Sellers. “Arithmetic Properties of Partitions with Even Parts Distinct.” *Ramanujan Journal* 23, no. 1–3 (2010): 169–81.
- Calkin, Neil, Nate Drake, Kevin James, Shirley Law, Philip Lee, David Penniston, and Jeanne Radder. “Divisibility Properties of the 5-regular and 13-regular Partition Functions.” *Integers: Electronic Journal of Combinatorial Number Theory* 8, no. 1 (2008).
- Dandurand, Brian, and David Penniston. “ ℓ -divisibility of ℓ -regular Partition Functions.” *Ramanujan Journal* 19, no. 1 (2009): 63–70.
- Eneström, Gustaf. *Verzeichnis der Schriften Leonhard Eulers*. Jahresbericht der Deutschen Mathematiker-Vereinigung 4. Leipzig, Germany: B. G. Teubner, 1910.
- Furcy, David, and David Penniston. “Congruences for ℓ -regular Partition Functions Modulo 3.” *Ramanujan Journal* 27, no. 1 (2012): 101–08.
- Gordon, Basil, and Ken Ono. “Divisibility of Certain Partition Functions by Powers of Primes.” *Ramanujan Journal* 1, no. 1 (1997): 25–34.
- Johansson, Fredrik. “Efficient Implementation of the Hardy-Ramanujan-Rademacher Formula.” *LMS Journal of Computation and Mathematics* 15 (2012): 341–59.
- Lovejoy, Jeremy. “Divisibility and Distribution of Partitions into Distinct Parts.” *Advances in Mathematics* 158, no. 2 (2001): 1–8.
- Ono, Ken. “The Web of Modularity.” *CBMS Regional Conference Series in Mathematics* (American Mathematical Society) no. 102 (2004): 18–19.
- Penniston, David. “Arithmetic of ℓ -regular Partition Functions.” *International Journal of Number Theory* 4, no. 2 (2008): 295–302.
- Ramanujan, Srinivasa. “Congruence Properties of Partitions.” In *Collected Papers of Srinivasa Ramanujan*, 232–38. Providence: AMS Chelsea, 2000.
- . “Some Properties of $p(n)$, the Number of Partitions of n .” *Proceedings of the Cambridge Philosophical Society* 19 (1919): 207–10.

Sturm, Jacob. "On the Congruence of Modular Forms." *Lecture Notes in Mathematics* 1240 (1987): 275–80.

Webb, J. J. "Arithmetic of the 13-regular Partition Function Modulo 3." *Ramanujan Journal* 25, no. 1 (2011): 49–56.

Appendix

Answers to Exercises

1. The partitions of $n = 4$ are:

- 4
- 3 + 1
- 2 + 2
- 2 + 1 + 1
- 1 + 1 + 1 + 1.

Thus, $p(4) = 5$.

2. The 3-regular partitions of $n = 5$ are:

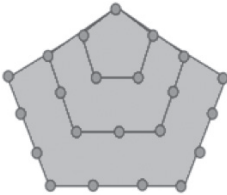
- 5
- 4 + 1
- 2 + 2 + 1
- 2 + 1 + 1 + 1
- 1 + 1 + 1 + 1 + 1.

Therefore, $b_3(5) = 5$.

3. $3(729n + 607) + 1 = (2187n + 1821) + 1 = 2187n + 1822$.

4. $3^{(5+2)}n + (5 \cdot 3^{(5+1)} - 1) / 2 = 2187n + (1644 / 2) = 2187n + 1822$.

5.



Counting the dots, we find that $G(4) = 22$.

6. $p(n) = p(n - 1) + p(n - 2) - p(n - 5) - p(n - 7) + p(n - 12) + p(n - 15) - - + \dots$

$p(5) = p(5 - 1) + p(5 - 2) - p(5 - 5) - p(5 - 7) + p(5 - 12) + p(5 - 15) - - + \dots$

$p(5) = p(4) + p(3) - p(0) - p(-2) + p(-7) + p(-10) - - + \dots$

$p(5) = p(4) + p(3) - p(0)$

$p(5) = 5 + 3 - 1$

$p(5) = 7$