Distinguishing Colored Links

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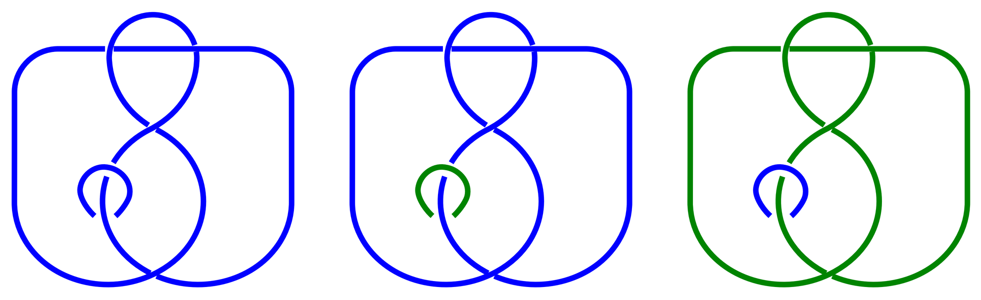
Colored Links

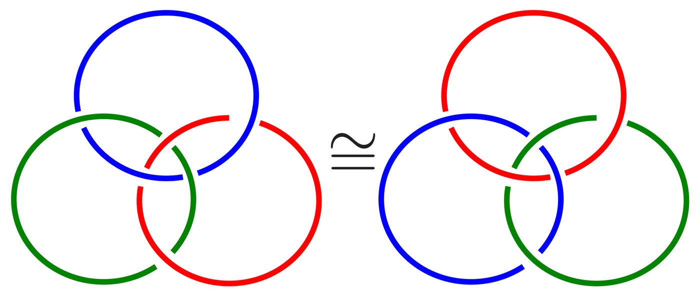
A colored link is a link where all of the components are colored. Below are three different colorings of the same link.



We've been studying whether changing the color changes the link, and how much information is gained by changing the colors of the components.

Main Question

Can we build examples of totally colored brunian links? We chose brunian links, because all the sublinks of a brunian link are the same, so you can't detect anything by sublinks. The Boromean Rings are brunian; however, they are not sufficient because they are colorblind.



Computations

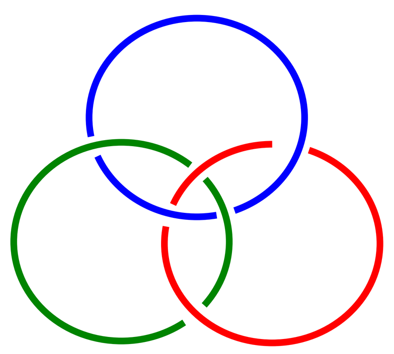
We used various computational tools as we tried to find our family of links. One such tool was the multivariable Alexander polynomial. Using a Maple implementation of an algorithm by H.R. Morton [1], we were able to quickly compute for the multivaraible Alexander polynomial of the braid by plugging the braid word into the program. Morton's theorem tells how to find the multivariable Alexander polynomial of a closed braid [2].

Theorem. *The multivariable Alexander polynomial* Δ_β *is given by the characteristic polynomial* $det(I - \bar{B}_\beta(t_1, ..., t_n))$.

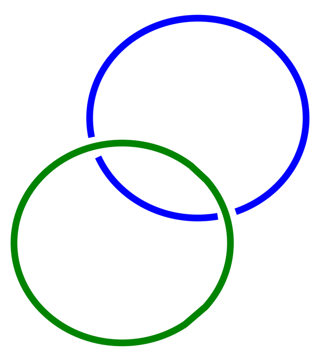
An example of the Burau matrix, $\bar{B}_\beta$, for σ_1 is $\begin{bmatrix} -a & 1 \\ 0 & 1 \end{bmatrix}$ where a is the color of the undercrossing.

Brunian Links

Brunian links are nontrivial links, meaning that when a component is removed, the rest become unlinked. Brunian links are an ideal starting point to study links. Any information gained from brunian links isn't coming from sublinks, since all the sublinks are the same. The boromean rings have this property.



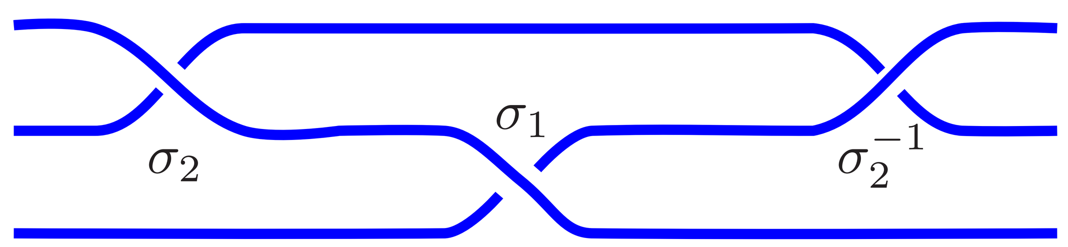
Boromean Rings



Boromean Rings Minus One Component

Braids

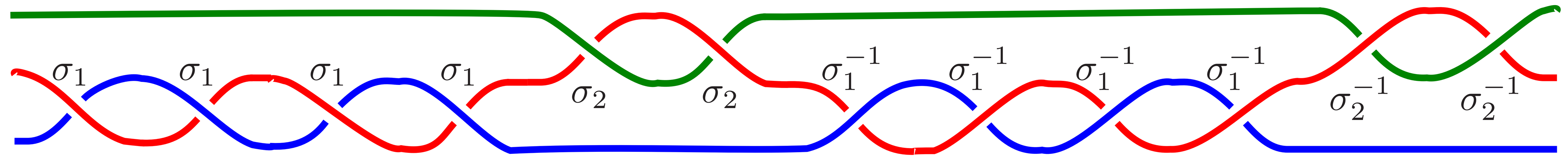
A braid is a collection of n strings in a solid cylinder in which the strings never turn around. Braids can be closed up to become links.
The braid word represents an n string braid by a word of σ_1 through σ_n.



This braid has a second crossing, a first crossing, and an inverse second crossing thus the resulting braid word is $\sigma_2\sigma_1\sigma_2^{-1}$.

Results

The braid has the following braid word. $\sigma_1\sigma_1\sigma_1\sigma_1\sigma_2\sigma_2\sigma_1^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_2^{-1}$



The braid word is plugged into the Maple program which gives the resulting polynomial.

$$\Delta_L = -(t_2 - 1)(t_1 - 1)(t_1 t_2 + 1)^2(t_3 - 1)$$

Theorem. *The results show that the red and blue components of the braid are interchangeable. However, the red component is not interchangeable with the green, since* $\Delta_L(t_1, t_2, t_3) \neq \Delta_L(t_1, t_3, t_2)$.

Bibliography

[1] H.R. Morton and Julian Hodgson. Maple procedure for calculating the multi-variable alexander polynomial. url-http://www.liv.ac.uk/ su14/programs /multiburau.maple/, 1996–1999.

[2] H. R. Morton. The multivariable Alexander polynomial for a closed braid. In *Low-dimensional topology (Funchal, 1998)*, volume 233 of *Contemp. Math.*, pages 167–172. Amer. Math. Soc., Providence, RI, 1999.

[3] Colin C. Adams. *The Knot Book*. American Mathematical Society, 2nd edition, 2004.

Strategy

1. We are studying braids of the form $\sigma_1^{2n}\sigma_2^{2m}\sigma_1^{-2n}\sigma_2^{-2m}$. The crossings are in even numbers to ensure that the result is a pure braid, the components end where they start.
2. Morton's computational tool is used to determine the multivariable Alexander polynomial of the braid. The resulting is polynomial is written down and categorized.
3. The number of crossings for the braid is changed, but the number of components is kept constant at three.

The procedure is repeated in the hope of furthering our research of finding an equation to prove our infinite family of completely colored, brunian links.

Future Research

We have been able to check whether each individual knot is colorblind; however, we have not been able to find a computation to prove all of them at the same time. We hope to find a general formula for the multivariable Alexander polynomial of these links.