Report 180

Feedforward as a Supplement to Feedback Adjustment in Allowing for Feedstock Changes

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Feedforward as a Supplement to Feedback Adjustment
in Allowing for Feedstock Changes

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Many industrial processes must be adjusted from time to time to continuously maintain their mean close to target. Compensations for deviations of the process mean from target may be accomplished by feedback and/or by feedforward adjustment. Feedback adjustments are made in reaction to errors at the output; feedforward adjustments are made to compensate anticipated changes. This article considers the complementary use of feedback and feedforward adjustments to compensate for anticipated step changes in the process mean as may be necessary in a manufacturing process each time a new batch of feedstock material is introduced. We consider and compare five alternative control schemes: (1) feedforward adjustment alone, (2) feedback adjustment alone, (3) feedback-feedforward adjustment, (4) feedback and indirect feedforward to increase the sensitivity of the feedback scheme, and (5) feedback with both direct and indirect feedforward.

KEY WORDS: Feedback control; Feedforward control; First-order dynamics; Nonstationary disturbance; Repeated adjustment scheme; Statistical process control.

1. INTRODUCTION

Traditionally methods for statistical process control (SPC) have had as one important objective the detection, and where possible, the assignment and removal of disturbances due to "special causes" (e.g., see Deming 1986). Methods of this kind have been highly successful because such continued debugging can produce steady improvement in the system by reduction of variation and simplification of operation. However, it is impossible for all process disturbances, even when
detectable, to be removed economically in this way. Indeed, as has been argued elsewhere (Box and Kramer 1992), the hypothesis of an unadjusted process remaining in a perfect state of statistical control contradicts the second law of thermodynamics and must be regarded therefore as a purely theoretical concept. This is borne out by careful study of the performance of a number of operating quality control schemes. For example, Alwan and Roberts (1988) found that process instability was common after standard techniques of quality control had been applied. Also it is noteworthy that in the highly successful Six Sigma system for process improvement (e.g., see Smith 1992, Harry 1994, Hoerl 1998, Box and Luceño 2000) additional allowance is made for a one and a half sigma drift of the local mean about the target value. It would be expected then that complementary methods based on process adjustment would be of value as an adjunct to the debugging process mentioned above, and recently there has been a resurgence of interest in the use of adjustment techniques suitable for application in the SPC context such as was discussed for example by Box and Jenkins (1970); MacGregor (1972); Fearn and Maris (1991); Vander Wiel, Tucker, Faltin, and Doganaksoy (1992); Jensen and Vardeman (1993); Tucker, Faltin, and Vander Wiel (1993); Box, Jenkins, and Reinsel (1994); Luceño, González, and Puig-Pey (1996); Box and Luceño (1997); Montgomery and Woodall (1997); and Luceño (2000).

Feedback adjustment occurs when compensatory changes are made in some suitable adjustment variable $X$ in reaction to output deviations from target. Feedforward adjustment occurs when such changes are made to compensate anticipated deviations. We consider here the situation where feedforward control might be used to compensate for expected level shifts. A common example is when adjustments are made to compensate for batch to batch differences based on pretest of the incoming raw material. However, such feedforward adjustment if used alone could prove inadequate because pretesting is subject to measurement errors. Thus even if the background disturbance was stationary, feedforward control used alone could result in systematic deviations from the process target. In this article therefore we consider to what extent is it helpful in the SPC context to use feedforward adjustment of one kind or another in addition to feedback adjustment.
1.1 A Simple System of Feedback Adjustment

We suppose that the output quality characteristic we desire to control is measured at equally spaced intervals of time. Thus $e_t$ is the output error observed at time $t$ for the system under control and $x_t = X_t - X_{t-1}$ is the consequent input adjustment made at time $t$. Then in what follows we consider a system of feedback control, sufficiently simple to be readily usable in the SPC context, that consists of repeatedly making an adjustment just sufficient to cancel a fixed proportion $G$ of the current output deviation from target.

The resulting feedback adjustment equation is then

$$g \ x_t = - G \ e_t,$$

(1.1)

where $0 < G \leq 1$ will be called the damping factor. The constant $g$ is the system gain, that is the eventual change in the output that is induced by a unit adjustment at the input. Appropriate values for $G$ are often in the range .1 to .4.

Although this type of feedback adjustment is of great simplicity, it has interesting characteristics, labeled below as A, B, and C, that are discussed more fully and justified in Box and Luceño (1997):

(A) The adjustment equation (1.1) is the discrete analog of continuous integral control because, by summing Equation (1.1), the overall correction in a period $t$ intervals long is

$$g \ X_t = g \ X_0 + g \ \Sigma_{i=1}^{t} x_i = g \ X_0 - G \ \Sigma_{i=1}^{t} e_i = k_0 + k_1 \ \Sigma_{i=1}^{t} e_i,$$

(1.2)

where $k_0 = g \ X_0$ and $k_1 = -G$ are constants.

(B) It further follows from Equation (1.1) that such control is equivalent to arranging that the overall correction is

$$g \ X_t = - \tilde{y}_t,$$

(1.3)

where

$$\tilde{y}_t = G \left[ y_t + H \ y_{t-1} + H^2 \ y_{t-2} + \ldots \right]$$

is an exponentially weighted moving average (EWMA) of the output deviations from target $y_t$, $y_{t-1}$, ...

... that would have occurred if no control had been applied. The constant $H = 1 - G$, in such an EWMA, is often called the smoothing constant or discount factor. Notice that Equations (1.2) and
(1.3) make no specific model assumptions about \( y_t, y_{t-1}, \ldots \) but follow from (1.1) using \( e_t = y_t + gX_{t-1} \). Equation (1.3) does show however that the adjustment equation (1.1) is intuitively reasonable since it is equivalent to employing \( \tilde{y}_t \) as a forecast made at time \( t \) for \( y_{t+1} \) and arranging that the total adjustment \( X_t \) just cancels this forecasted deviation. The effect of doing this is to replace the deviation \( y_t \) that would occur at the output if there were no control, by its error of forecast \( y_{t+1} - \tilde{y}_t = e_{t+1} \).

(C) The use of the adjustment equation (1.1) may be formally justified using the following assumptions:

(i) The uncontrolled output \( y_t \) may be represented by the integrated moving average (IMA) time series model

\[
y_t - y_{t-1} = a_t - \theta a_{t-1},
\]

where \( \{a_t\} \) is a "white noise" sequence; that is, a series of independent and identically distributed (iid) random variables having mean 0 and standard deviation \( \sigma_a \).

(ii) The system is responsive, that is to say that the dynamics are such that the adjustment \( x_t \) produces its full effect at the output in one time period.

With these assumptions, the output errors will have zero mean and, if \( G \) is set equal to \( 1 - \theta = \lambda \) (so that \( H = \theta \)), the adjustment scheme will produce minimum mean squared error (MMSE) at the output and so will minimize the output variance \( \text{var}(e_t) \). In that case, the variance of the adjustments will satisfy \( g^2 \text{var}(x_t) = G^2 \text{var}(e_t) \) with \( \text{var}(e_t) = \sigma_a^2 \).

In practice, it is frequently advantageous to employ a value for \( G \) less than \( \lambda \) (i.e., \( H > \theta \)) for which the needed adjustments will be smaller in magnitude than those yielding MMSE (e.g., see Box and Luceño 1995). This yields a constrained adjustment scheme which minimizes \( \text{var}(e_t) \) subject to a given reduction in \( g^2 \text{var}(x_t) = G^2 \text{var}(e_t) \). Such constrained schemes are attractive since large reductions in \( \text{var}(x_t) \) can frequently be obtained at the cost of only small increases in \( \text{var}(e_t) \). For example, suppose that the system gain \( g = 1 \) and \( \theta = .6 \) in Equation (1.4) so that \( \lambda = 1 - \theta = .4 \) and the MMSE scheme is \( x_t = - .4 e_t \), but if instead of setting \( G = .4 \) we used the value
\[ G = 0.2 \ (H = 0.8) \], then it is easy to show that the standard deviation of \( x_t \) is reduced by 52.7% at the expense of an increase in the standard deviation of \( e_t \) of only 5.4%.

In favor of the assumption that the uncontrolled series may, to an acceptable approximation, be represented by the IMA of equation (1.4) are the following:

(i) formal time series analysis of production data has frequently shown this to be an adequate model,

(ii) with this model the variance of the difference between observations \( m \) steps apart is a linear function of \( m \) so that the variogram for this time series model increases linearly with \( m \) (see Box and Luceño 1997, chap. 12), and thus can be explained by the intuitively reasonable concept of "sticky innovations" discussed by Box and Kramer (1992),

(iii) Adjustment using equation (1.3) appears to be extremely robust, that is to say, it still can provide effective control in conditions widely differing from the ideal. Specifically, Box and Luceño (1997) demonstrate such robustness when contrary to assumption: there is inertia in the system response approximated by first-order dynamics, and/or there are one or two intervals of pure delay in the system response, or the distribution of errors deviates in likely ways from normality, or the correct time series model is an autoregressive process rather than an IMA process (see also Luceño 1998).

1.2 Allowance for Expected Step Changes

In this paper we consider the commonly occurring problem of how to compensate for expected step changes in the output that typically occurs in a manufacturing process whenever new batches of feedstock material are introduced. Five methods of adjustment for compensating such step changes are:

Scheme 1: Feedforward adjustment alone.

Scheme 2: Feedback adjustment alone.

Scheme 3: Feedback-Feedforward adjustment.

Scheme 4: Feedback and indirect feedforward to increase the sensitivity of feedback.
1.3 Transient Produced by Feedback Control

As a preliminary it is of interest to determine the time that would elapse before the feedback adjustment equation (1.1) would reduce a step change to a negligible value.

For discrete data, a unit step change made at time 0 may be represented by a series consisting of "zeros" for \( t < 0 \) followed by "ones" for \( t \geq 0 \). Adjustment of such a series by Equation (1.1) produces for \( t \geq 0 \) the geometric series 1, \( H, H^2, \ldots \).

This corresponds in continuous time to the exponential decay of the impulse-response function \( w(t) = (g / \tau) \exp(-t / \tau) \) of a first-order linear dynamic system, where the constant \( \tau \) is called the time constant of the system. This is the time at which the effect of a step change is reduced to a proportion \( e^{-1} = .368 \) (i.e., 36.8%) of its initial value. The smoothing constant of the first-order system sampled at equispaced unit times satisfies \( H^\tau = e^{-1} \) so that \( \tau = -1 / \ln H \). In particular, \( 3\tau \) is a convenient benchmark because this is the approximate time at which the response would decay to about 5% of its initial value. For example, for \( G = .4, \; \tau = 1.96 \) and \( 3\tau = 5.87 \). The closest integer is 6 and gives \( H^6 = .047 \) (i.e., 4.7%), which is close to the desired value of 5%.

For the control schemes described below, we will assume that at time \( t \) the deviation from target \( y_t \) that would occur if there were no interventions can be represented by the IMA time series model (1.4). We suppose that a batch of feedstock lasts for \( T \) intervals and that an event occurring in the \( i \)th interval of the \( h \)th batch is indexed by \( t = i + hT \). Interest centers on the size of the output errors \( e_t \) (measured as deviations from target after control) and the corresponding adjustments \( x_t \) that are needed, which are measured as deviations from zero. We consider the mean squared deviations of each of these quantities. When the target values and the means coincide, then mean squared deviations are the variances \( \text{var}(e_t) \) and \( \text{var}(x_t) \). By setting \( \sigma_a^2 = 1 \), all variances can be represented as multiples of the noise variance \( \sigma_n^2 \).
Section 2 compares the characteristics of the five schemes of Section 1.2. In Sections 3 and 4 we consider the optimal choice of the damping factor(s). Concluding remarks are given in Section 5.

2. CHARACTERISTICS OF THE SUGGESTED SCHEMES

For the uncontrolled process over infinite time, the variance of \( y_t \) in (1.4) is infinite. However, for a finite period of length \( T \), the average value is
\[
M_0 = T^{-1} \sum_{i=1}^{T} \text{var}(y_i) = 1 + (T - 1) \lambda^2 / 2.
\]

2.1 Scheme 1: Feedforward Alone (FF)

We will denote by \( \mu_t \) the step change in the \( t \)th batch and we suppose that, from batch to batch, \( \mu_t \) has mean 0 and variance \( \sigma_{\mu}^2 \). We also assume that the step changes \( \mu_t \) can be estimated by pretest with measurement errors \( \epsilon_i \) having mean 0 and variance \( \sigma_{\epsilon}^2 \). We denote by \( m_t = \mu_t + \epsilon_t \) the observed value of \( \mu_t \) and by \( \sigma_m^2 \) the variance of \( m_t \).

Then the total disturbance \( z_t \) resulting from the combination of Equation (1.4), for the background disturbance \( y_t \), and the step change \( \mu_t \) is
\[
z_t = z_{t-1} + \theta a_{t-1} + \mu_t,
\]
where \( \mu_t = 0 \) for any \( t \neq 0, T, 2T, 3T, \ldots \). The uncontrolled process would now have variances
\[
M_0 = T^{-1} \sum_{i=1}^{T} \text{var}(z_i) = 1 + (T - 1) \lambda^2 / 2 + \sigma_{\mu}^2
\]
and the effect of the feedforward control will be to reduce this to
\[
M_1 = 1 + (T - 1) \lambda^2 / 2 + \sigma_{\epsilon}^2.
\]

Because only one adjustment \( g x_{t-1} = -m_t \) to compensate for \( m_t \) is made for each batch of length \( T \), the required adjustments satisfy \( V_1 = T^{-1} \sum_{i=1}^{T} \text{var}(g x_i) = \sigma_m^2 / T \).

2.2 Scheme 2: Feedback Alone (FB)

Suppose now that the process is controlled using only the feedback adjustment equation (1.1) with \( G \leq \lambda \). If there are no changes in mean from batch to batch, the errors at the output would
satisfy \( e_t - H e_{t-1} = a_t - \theta a_{t-1} \) (e.g., see Box and Luceño 1997, sec. 12.11) so that var \((e_t)\) would be given by
\[
\sigma_e^2 = \frac{1 - 2 \theta H + \theta^2}{1 - H^2}
\] (2.2)
and var \((g x_t)\) = \(G^2 \sigma_e^2\). In particular, if \(G = \lambda\) (or, equivalently, \(H = \theta\)), then var \((e_t)\) = \(\sigma_a^2 = 1\), the smallest value obtainable.

Suppose now that the current batch contributes a deviation in mean \(\mu_t\) lasting for \(T\) intervals and that \(T\) is large compared with the time constant of the system (so that \(H^2 T \equiv 0\)). Then the total disturbance \(z_t\) will be given by (2.1), and the error at the output \(e_t\) at time \(t\) will be the sum of the total disturbance \(z_t\) and the accumulated effects of the compensations \(g X_{t-1}\) performed up to time \(t-1\), that is,
\[
e_t = z_t + g X_{t-1}.
\] (2.3)
Combining Equations (1.1), (2.1), and (2.3), one obtains \(e_t - H e_{t-1} = a_t - \theta a_{t-1} + \mu_t\), so that \(E(e_t) = 0\) and the long-run mean squared error at the output (MSEO) is given by
\[
M_2 = \frac{1}{T} \sum_{i=1}^{T} \text{var}(e_i) = \sigma_e^2 + \frac{\sigma_\mu^2}{T (1 - H^2)}.
\]
where \(\sigma_e^2\) is given by (2.2). To obtain these errors at the output, adjustments according to (1.1) must be made at the input so that the long-run input variance is then \(V_2 = T^{-1} \sum_{i=1}^{T} \text{var}(g x_i) = G^2 M_2\).

2.3 Scheme 3: Feedback Plus Feedforward (FB+FF)

If we now use feedforward to compensate the batch mean \(m_t\) observed at time \(t-1\) with a measurement error having variance \(\sigma_e^2\), the adjustment equation (1.1) should be replaced by \(g x_{t-1} = -G e_{t-1} - m_t\), where \(m_t = 0\) for \(t \neq 0, T, 2T, 3T, \ldots\). Consequently, the errors at the output follow the recursive relation \(e_t - H e_{t-1} = a_t - \theta a_{t-1} + \epsilon_t\), the expected error is null, and the MSEO is given by
\[
M_3 = \sigma_e^2 + \frac{\sigma_\epsilon^2}{T (1 - H^2)}.
\]
The long-run variance at the input is \(V_3 = G^2 M_3 + \sigma_m^2 / T\).
2.4 Scheme 4: Feedforward to Increase the Sensitivity of the Feedback System (FB+FFS)

Suppose now that when a new batch is to be initiated there is no direct feedforward. Instead, the information concerning the time when a change will occur is fed forward to temporally increase the sensitivity of the feedback system. Specifically, the sensitivity of the feedback control is temporally increased by using a larger damping factor $\tilde{G}$ in the feedback adjustment equation during the first $k$ intervals for each new batch, and $k \ll T$ is chosen so that after $k$ intervals the effect of the transient has decayed to a negligible value. Then the adjustment equations for the batch starting at time $t = T$ are $g x_t = -G e_t$ (for $t < T$), $g x_t = -\tilde{G} e_t$, (for $t = T, \ldots, T + k - 1$), and $g x_t = -G e_t$, (for $t = T + k, \ldots, 2T - 1$). The errors at the output will follow the recursive relations

\begin{align}
e_t - H e_{t-1} &= a_t - \theta a_{t-1}, \quad (t < T) \\
e_T - H e_{T-1} &= a_T - \theta a_{T-1} + \mu_T, \quad (2.4a) \\
e_t - H e_{t-1} &= a_t - \theta a_{t-1}, \quad (t = T + 1, \ldots, T + k) \quad (2.4b) \\
e_t - H e_{t-1} &= a_t - \theta a_{t-1}, \quad (t = T + k + 1, \ldots, 2T - 1) \quad (2.4c) \\
e_{2T} - H e_{2T-1} &= a_{2T} - \theta a_{2T-1} + \mu_{2T}, \quad (2.4d)
\end{align}

and so on, where $\tilde{H} = 1 - \tilde{G}$. Again $E(e_t) = 0$. Assuming that $T$ is large enough so that $H^{2T} \equiv 0$, $\tilde{H}^{2T} \equiv 0$, and $k \ll T$, the MSEO and input variances are given by

\begin{align}M_4 &= \frac{k}{T} \frac{\sigma^2_e}{\sigma^2_e} + \frac{T - k}{T} \frac{\sigma^2_e}{\sigma^2_e} + \frac{\sigma^2_e}{\sigma^2_e} \left( \tilde{G} \tilde{H}^{2k} + \tilde{G} \left( 1 - \tilde{H}^{2k} \right) \right) + \frac{\xi}{T} \left( \frac{1}{1 - H^2} - \frac{1}{1 - \tilde{H}^2} \right) \\
\end{align}

and

\begin{align}V_4 &= \frac{k}{T} \frac{\sigma^2_e}{\sigma^2_e} \tilde{G}^2 + \frac{T - k}{T} \frac{\sigma^2_e}{\sigma^2_e} \tilde{G}^2 + \frac{\sigma^2_e}{\sigma^2_e} \left( \frac{G \tilde{H}^{2k}}{2 - G} + \tilde{G} \left( 1 - \tilde{H}^{2k} \right) \right) + \frac{\xi}{T} \left( \frac{G}{2 - G} - \tilde{G} \right), \end{align}

respectively, where $\sigma^2_e$ is given by (2.2),

\begin{align}\sigma^2_e = \frac{1 - 2 \theta \tilde{H} + \theta^2}{1 - \tilde{H}^2}, \quad \text{(2.5)}
\end{align}

and

\begin{align}\xi = \left( \frac{\sigma^2_e}{\sigma^2_e} - \sigma^2_e \right) \left( 1 - \tilde{H}^{2k} \right). \quad \text{(2.6)}
\end{align}
The Appendix provides methods to compute these long-run MSEO and input variances that apply for responsive and first-order dynamics.

2.5 Scheme 5: Feedback With Both Direct and Indirect Feedforward (FB + FF + FFS)

In addition to feedback, feedforward is used to cancel the measured batch mean and also to temporally increase the sensitivity of the feedback adjustment. The adjustment equations for the batch starting at time \( t = T \) are as in Section 2.4 excepting that now \( g x_{T-1} = -G e_{T-1} - m_T \), \( g x_{2T-1} = -G e_{2T-1} - m_{2T} \), and so on.

Consequently, noting that \( \mu_T = m_T - e_T \) in Equation (2.4b), the term containing \( m_T \) disappears from (2.4b) and hence the effect of \( m_T \) on the MSEO, \( M_5 \), is zero. The variation associated with \( \mu_T \) is passed to the input variance \( V_5 \), which now contains a new term of magnitude \( \sigma_m^2 / T \).

Consequently, for large \( T \),

\[
M_5 = \frac{k}{T} \hat{\omega}^2 + \frac{T - k}{T} \sigma_e^2 + \frac{\sigma_{e_k}^2}{T} \left( \frac{\tilde{H}^{2k}}{1 - H^2} + \frac{1 - \tilde{H}^{2k}}{1 - H^2} \right) + \frac{\xi}{T} \left( \frac{1}{1 - H^2} - \frac{1}{1 - H^2} \right)
\]

and

\[
V_5 = \frac{k}{T} \hat{\omega}_e^2 G^2 + \frac{T - k}{T} \sigma_e^2 G^2 + \frac{\sigma_{e_k}^2}{T} \left( \frac{G \tilde{H}^{2k}}{2 - G} + \frac{\tilde{G}(1 - \tilde{H}^{2k})}{2 - \tilde{G}} \right) + \frac{\sigma_m^2}{T} + \frac{\xi}{T} \left( \frac{G}{2 - G} - \frac{\tilde{G}}{2 - \tilde{G}} \right)
\]

where \( \sigma_e^2 \), \( \hat{\omega}_e^2 \), and \( \xi \) are given by (2.2), (2.5), and (2.6), respectively.

2.6. Relative Performance of the Five Schemes.

Comparison of the various schemes is made easier by considering close approximations to the exact expressions derived earlier. These are shown in Table 1.

It is important to notice that the terms in the table for \( M_i \) (\( i = 1, \ldots, 5 \)) that change from one scheme to another can be of order \( T \), \( T^0 \), or \( T^{-1} \) (the terms of order \( T^0 \) are enclosed by square brackets in Table 1). Thus,
a) Unless $T$ is small, the schemes 2, 3, 4, and 5 that include feedback will yield markedly smaller values of $M_1$ than the pure feedforward scheme 1 because they eliminate the nonstationary term of order $T$ that is present in $M_1$.

b) By augmenting feedback control with feedforward as in scheme 3 compared with scheme 2, or in scheme 5 compared with scheme 4, the batch to batch variance $\sigma_\mu^2$ is replaced by the measurement variance $\sigma_\varepsilon^2$; however the terms involved are only of order $T^{-1}$.

c) By appropriate choice of $k$, $G$, and $\tilde{G}$, the MSEO in schemes 4 and 5 can always be made smaller than those in schemes 2 and 3, respectively.

(Please place Tables 1 and 2 near here.)

For given estimates of $\sigma_\mu$, $\sigma_\varepsilon$, and $\lambda$, using these expressions, the values of $M_1$ and $V_i$ for the various schemes may be calculated and compared. The results should be interpreted in the light of practical necessities and in particular the convenience of running the different schemes. To illustrate these calculations, we consider the following example. Suppose that the values of the parameters are $T = 100$, $\theta = .8$ ($\lambda = .2$), $\sigma_\alpha = 1$, $\sigma_\mu = 2$, $\sigma_\varepsilon = .5$, $G = .2$, $\tilde{G} = .8$, and $k = 3$. Thus there is a 25% error in the measurements of the batch means. From Table 2 we see that

(i) Because the background disturbance is nonstationary, without feedback control to eliminate this nonstationary disturbance the MSEO, $M_1$, will be large.

(ii) The MSEO $M_2$ is greatly reduced because the nonstationarity is eliminated by feedback adjustment. However, it still contains a component depending on $\sigma_\mu$ because of the transient in the response to feedback. The input variances are almost the same as in scheme FF; the main difference is that scheme FB makes small adjustments every time, whereas only one big adjustment at each time of batch change is called for by scheme FF.

(iii) A further reduction in MSEO (from $M_2$ to $M_3$) is obtained by eliminating the measured batch changes by feedforward. However, a transient due to measurement errors still remains. The input variance $V_3$ is almost twice $V_2$. 
(iv) The addition of indirect (rather than direct) feedforward to increase feedback sensitivity produces in this case a MSEO $M_4$ smaller than $M_2$ but larger than $M_3$. Because the damping factor $\tilde{G}$ has not been chosen carefully, the required input variance $V_4$ is larger than $V_2$ and $V_3$.

(v) The addition of direct and indirect feedforward produces MSEO $M_5$ and input variance $V_5$ larger than $M_3$ and $V_3$, respectively, again because of the choice of $G$ and $\tilde{G}$.

3. OPTIMAL VALUES FOR THE DAMPING FACTORS

In Table 2, the fact that $M_5$ is larger than $M_3$ deserves further consideration. In scheme 3, there is one free parameter ($G$) to be chosen arbitrarily, whereas in scheme 5 there are two such parameters ($G$ and $\tilde{G}$). When the free parameters $G$ and, possibly, $\tilde{G}$ are chosen to minimize $M_i$, appropriate choices can always be made to yield a value of $M_5$ smaller than $M_3$. Table 3 illustrates the relative performance of the schemes previously considered. The values of the parameters in Table 3 are again $T = 100$, $\theta = .8$ ($\lambda = .2$), $\sigma_\epsilon = 1$, $\sigma_\mu = 2$, $\sigma_\theta = .5$, and $k = 3$, but $G$ and $\tilde{G}$ are now chosen to minimize the MSEO. We see that

(i) The MSEO may be reduced somewhat by choosing $G$ and $\tilde{G}$ adequately.

(ii) The MSEO produced by feedback control $M_2$ may be reduced to $M_3$ by adding direct feedforward, without increasing the average input variance appreciably.

(iii) Indirect feedforward is less efficient than direct feedforward in reducing the MSEO.

(iv) The scheme using direct and indirect feedforward provides values for $M_5$ and $V_5$ that are slightly smaller than $M_3$ and $V_3$ for direct feedforward, but the latter may be simpler to use.

(Please place Tables 3 and 4 near here.)

4. COMPARISON OF OUTPUT MEAN SQUARED ERRORS WITH INPUT VARIANCE FIXED

A difficulty in comparing schemes 2, 3, 4, and 5 is that we cannot be guided by the values of $M_i$ alone for, as we see in Tables 2 and 3, these can be associated with very different values of the required input variances $V_i$. To overcome this difficulty, constrained feedback adjustment schemes may be explored that minimize the MSEO with respect to $G$, and possibly $\tilde{G}$, subject to the
restriction of the input variance \( \text{INPVAR} \leq \gamma \), where \( \gamma \) is some chosen constant. (For the particular cases in which \( \gamma \) is very large, unconstrained adjustment schemes such as those of Table 3 are obtained.)

The input variances may be reduced considerably by using constrained adjustment schemes provided that a modest increase in the MSEO may be accepted. In Table 4 therefore we have calculated the values of \( M_i \) when \( V_i \) have the same fixed bounds. Three examples are given with the input variances bounded by \( \gamma = .06, .0525, \) and \( .045 \). We see that the schemes FB+FF and FB+FF+FFS provide almost the same overall performance, which is better than the performance of the other schemes when \( \gamma = .06 \). However, as the constraint in the input variance is more important (i.e., as \( \gamma \) decreases), the performance of the scheme FB+FFS improves in comparison with the other schemes and is the best for \( \gamma = .045 \).

5. CONCLUDING REMARKS

The comparison among the five schemes of Section 1.2 may be summarized as follows:

(i) Because feedforward adjustment alone cannot eliminate nonstationary disturbances, and as in any case based on dead reckoning, any reasonable form of feedback control is likely to do much better.

(ii) If scheme 2 employing simple feedback is used as a basis for comparison, some improvement in mean squared error at the output is possible by use of direct and/or indirect feedforward to supplement the feedback scheme.

(iii) For unconstrained schemes, the use of direct feedforward to complement feedback is better than simple feedback alone but, for highly constrained schemes it does less well.

(iv) For constrained schemes, the sensitized feedforward scheme to complement feedback outperforms the other schemes.

ACKNOWLEDGMENTS

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APPENDIX. LONG-RUN MEAN SQUARED ERROR AT THE OUTPUT AND INPUT VARIANCE FOR THE SCHEMES OF SECTION 2 UNDER FIRST-ORDER DYNAMICS

Scheme "4" with first-order dynamics. For generality, we consider the evaluation of the long-run mean squared error at the output and input variance under first-order dynamics and proportional integral (PI) control. The PI adjustment equation is

\[ g x_t = -G[(1 + P) - PB]e_t; \]  \hspace{1cm} (A.1)

where \( G \) and \( P \) are constants. The error at the output is

\[ e_t = z_t + \frac{g(1 - \delta)}{1 - \delta B} X_{t-1}, \]

where \( B \) is the backshift operator such that \( BX_t = X_{t-1} \), \( g \) is the system gain and \( \delta \) is the inertia. The total disturbance is given by

\[ z_t - z_{t-1} = a_t - \theta a_{t-1} + \mu_t, \]

where \( \mu_t = 0 \) for any \( t \neq T, 2T, 3T, \ldots \).

For \( t < T \), the errors at the output satisfy

\[ e_t = a_t - \phi_1 a_{t-1} - \phi_2 a_{t-2} + \phi_1 e_{t-1} + \phi_2 e_{t-2}, \] \hspace{1cm} (A.2)

where \( \phi_1 = 1 + \delta - G(1 + P)(1 - \delta), \phi_2 = -\delta + G P(1 - \delta) \), \( \theta_1 = \theta + \delta \), and \( \theta_2 = -\theta \delta \). Similarly, for \( t = T \), we have

\[ e_T = a_T - \phi_1 a_{T-1} - \phi_2 a_{T-2} + \phi_1 e_{T-1} + \phi_2 e_{T-2} + \mu_T. \] \hspace{1cm} (A.3)

Equation (A.3) shows that \( \mu_T \) affects the error at the output \( e_T \) at time \( T \) so that we temporally increase the sensitivity of the feedback scheme by replacing (A.1) with

\[ g x_t = -\tilde{G}[(1 + \tilde{P}) - \tilde{PB}]e_t \hspace{0.5cm} (t = T, \ldots, T + k - 1). \] \hspace{1cm} (A.4)

Then, for \( T + 1 \leq t \leq T + k \), the errors at the output satisfy

\[ e_t = a_t - \tilde{\phi}_1 a_{t-1} - \tilde{\phi}_2 a_{t-2} + \tilde{\phi}_1 e_{t-1} + \tilde{\phi}_2 e_{t-2}, \]

where \( \tilde{\phi}_1 = 1 + \delta - \tilde{G}(1 + \tilde{P})(1 - \delta) \) and \( \tilde{\phi}_2 = -\delta + \tilde{G} \tilde{P}(1 - \delta) \). Finally, for \( T + k < t \leq 2T \), the underlying PI scheme is resumed so that the errors at the output satisfy (A.2) for \( T + k < t < 2T \) and (A.3) with \( T \) replaced by \( 2T \), and so on.
From (A.2), the expressions for \( \zeta_0 = \text{var}(e_t) \) and \( \zeta_1 = \text{cov}(e_t, e_{t-1}) \), for \( t < T \), can be obtained by using standard time series methods (e.g., see Luceño 1993). Assuming for the moment that \( \mu_T = 0 \), Equations (A.2) and (A.3) can be used to obtain

\[
\text{var}(e_T) = (1 + v' A_T v) \sigma_a^2, \\
\text{var}(e_{T+1}) = (1 + \tilde{v}' A_{T+1} \tilde{v}) \sigma_a^2, \\
\text{var}(e_t) = (1 + \tilde{v}' \tilde{A}_t \tilde{v}) \sigma_a^2, \quad (T + 2 \leq t \leq T + k) \\
\text{var}(e_{T+k+1}) = (1 + v' \tilde{A}_{T+k+1} v) \rho_a^2, \\
\text{var}(e_t) = (1 + v' A_t v) \sigma_a^2, \quad (T + k + 2 \leq t \leq 2T),
\]

and so on, where \( v' = (-\theta_1, -\theta_2, \phi_1, \phi_2) \) and \( \tilde{v}' = (-\tilde{\theta}_1, -\tilde{\theta}_2, \tilde{\phi}_1, \tilde{\phi}_2) \); and \( \sigma_a^2 A_t \) and \( \sigma_a^2 \tilde{A}_t \) stand for the covariance matrix of vector \((a_{t-1}, a_{t-2}, e_{t-1}, e_{t-2})\) depending on whether this matrix is a function of \( \phi_1 \) and \( \phi_2 \) or of \( \tilde{\phi}_1 \) and \( \tilde{\phi}_2 \). Thus for \( t = T \) or \( T + 1 \),

\[
A_t = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & \alpha_t & 1 \\ 1 & \alpha_t & \beta_t & \gamma_t \\ 0 & 1 & \gamma_t & \delta_t \end{pmatrix},
\]

where \( \alpha_T = \phi_1 - \theta_1 \), \( \delta_T = \zeta_0 / \sigma_a^2 \), \( \gamma_T = \zeta_1 / \sigma_a^2 \), and \( \beta_T = \delta_T \), and then

\[
\alpha_{T+1} = \phi_1 - \theta_1, \quad \delta_{T+1} = \beta_T, \quad \gamma_{T+1} = -\theta_1 - \theta_2 \alpha_T + \phi_1 \beta_T + \phi_2 \gamma_T, \quad (A.5a)
\]

\[
\beta_{T+1} = 1 - \theta_1 \alpha_{T+1} - \theta_2 (-\theta_2 + \phi_1 \alpha_T + \phi_2) + \phi_1 \gamma_{T+1} + \phi_2 (-\theta_2 + \phi_1 \gamma_T + \phi_2 \delta_T). \quad (A.5b)
\]

Similarly, for \( T + 2 \leq t \leq T + k + 1 \),

\[
\tilde{A}_t = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & \tilde{\alpha}_t & 1 \\ 1 & \tilde{\alpha}_t & \tilde{\beta}_t & \tilde{\gamma}_t \\ 0 & 1 & \tilde{\gamma}_t & \tilde{\delta}_t \end{pmatrix},
\]

where \( \tilde{\alpha}_t, \tilde{\delta}_t, \tilde{\gamma}_t, \) and \( \tilde{\beta}_t \) are computed using (A.5) where \( \phi_1 \) and \( \phi_2 \) are replaced by \( \tilde{\phi}_1 \) and \( \tilde{\phi}_2 \) and \( T + 1 \) by \( t \). Finally, the long-run mean squared error at the output (without taking the effect of \( \mu_T \) into account for the moment) is computed as \( M_{41} = T^{-1} \sum_{t=T}^{2T-1} \text{var}(e_t) \), where \( T \) is assumed to be large enough for the effect of \( \mu_T \) to settled down before the next step change.

The corresponding \( V_{41} \) computed without taking the effect of \( \mu_T \) into account (for the moment) may be computed using (A.1) and (A.4). Thus
\[
\text{var}(g x_t) = G^2 \left[ (1 + \tilde{P})^2 \beta_{t+1} + \tilde{P}^2 \delta_{t+1} - 2 \tilde{P} \left( 1 + \tilde{P} \right) Y_{t+1} \right] \sigma_a^2, \quad (t = T, \ldots, T + k - 1), \\
\text{var}(g x_T) = G^2 \left[ (1 + P)^2 \beta_{t+1} + P^2 \delta_{t+1} - 2 P \left( 1 + P \right) Y_{t+1} \right] \sigma_a^2, \quad (t = T + k, \ldots, 2T - 1), 
\]

and \( V_{41} = T^{-1} \sum_{t=T}^{2T-1} \text{var}(g x_t) \).

The effect of \( \mu_T \) is given by \( M_{42} = \sigma^2 \mu_T^{-1} \sum_{t=0}^{T-1} \omega_t^2 \), where \( \omega_0 = 0 \), \( \omega_0 = 1 \), \( \omega_t = \phi_1 \omega_{t-1} + \phi_2 \omega_{t-2} \quad (t = 1, \ldots, k) \), and \( \omega_t = \phi_1 \omega_{t-1} + \phi_2 \omega_{t-2} \quad (t > k) \). Correspondingly, \( V_{42} = T^{-1} \sum_{t=T}^{2T-1} \text{var}_2(g x_t) \) where

\[
\text{var}_2(g x_t) = \left[ -G \left( 1 + \tilde{P} \right) \pi_{t-T} + \tilde{G} \tilde{P} \pi_{t-T-1} \right] \sigma^2 \mu, \quad (t = T, \ldots, T + k - 1), \\
\text{var}_2(g x_T) = \left[ -G \left( 1 + P \right) \pi_{t-T} + GP \pi_{t-T-1} \right] \sigma^2 \mu, \quad (t = T + k, \ldots, 2T - 1).
\]

Finally, the long-run mean squared error at the output \( M_4 \) is the sum of \( M_{41} \) and \( M_{42} \), and the long-run input variance \( V_4 \) is the sum of \( V_{41} \) and \( V_{42} \).

**Scheme 
"2" with first-order dynamics.** The formulas for scheme "2" producing no augmentation of the feedback system can be obtained by substituting \((G, P)\) for \((\tilde{G}, \tilde{P})\). Assuming that \( T \) is large enough for the effect of \( \mu_T \) to settled down before the next step change, we obtain

\[
M_2 = \xi_0 + \frac{\sigma^2 \mu}{T} \left( 1 - \phi_2 \right) \frac{1}{(1 - \phi_2)^2 - \phi_1^2},
\]

and, similarly,

\[
V_2 = G^2 \xi_0 \left[ 1 + 2P(1 + P)(1 - \xi_1 / \xi_0) \right] \\
+ \frac{\sigma^2 \mu}{T} \frac{G^2}{1 + \phi_2} \frac{1}{(1 - \phi_2)^2 - \phi_1^2} \left[ 1 - \phi_2 \right] \left[ (1 + P)^2 + P^2 \right] - 2\phi_1 P(1 + P)
\]

**Scheme "3" with first-order dynamics.** The formulas for scheme "3" using feedforward to compensate directly for the measurable part of the input change can be obtained similarly, so that

\[
M_3 = \xi_0 + \frac{\sigma^2 \varepsilon}{T} \left( 1 - \phi_2 \right) \frac{1}{(1 - \phi_2)^2 - \phi_1^2},
\]

and

\[
V_3 = \sigma^2 \varepsilon \left[ (1 - \delta)^2 / T + G^2 \xi_0 \left[ 1 + 2P(1 + P)(1 - \xi_1 / \xi_0) \right] \right] \\
+ \frac{\sigma^2 \varepsilon}{T} \frac{G^2}{1 + \phi_2} \frac{1}{(1 - \phi_2)^2 - \phi_1^2} \left[ 1 - \phi_2 \right] \left[ (1 + P)^2 + P^2 \right] - 2\phi_1 P(1 + P)
\]

Page 16
Scheme "5" with first-order dynamics. Compute $M_4$ and $V_4$ using the methods for scheme "4" but replacing $\sigma^2_\mu$ by $\sigma^2_\xi$. Then take $M_5 = M_4$ and $V_5 = V_4 + \sigma^2_m \left( 1 + \delta^2 \right) (1 - \delta)^2 / T$.

REFERENCES


### TABLE 1a

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<tr>
<th>Scheme</th>
<th>Long-run mean squared errors at the output</th>
</tr>
</thead>
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<tr>
<td>1) FF</td>
<td>$M_1 = \frac{T-1}{2} \lambda^2 + \left[ 1 + \sigma_\epsilon^2 \right]$</td>
</tr>
<tr>
<td>2) FB</td>
<td>$M_2 = \left[ \frac{1-2 \theta H + \theta^2}{1-H^2} \right] + \frac{\sigma_\mu^2}{T \left( 1-H^2 \right)}$</td>
</tr>
<tr>
<td>3) FB+FF</td>
<td>$M_3 = \left[ \frac{1-2 \theta H + \theta^2}{1-H^2} \right] + \frac{\sigma_\epsilon^2}{T \left( 1-H^2 \right)}$</td>
</tr>
<tr>
<td>4) FB+FFS</td>
<td>$M_4 \equiv \left[ \frac{k}{T} \frac{1-2 \theta \tilde{H} + \theta^2}{1-\tilde{H}^2} + \frac{T-k}{T} \frac{1-2 \theta H + \theta^2}{1-H^2} \right] + \frac{\sigma_\mu^2}{T \left( 1-H^2 + 1-\tilde{H}^2 \right)} + \frac{\sigma_\epsilon^2}{T \left( 1-H^2 + 1-\tilde{H}^2 \right)}$</td>
</tr>
<tr>
<td>5) FB+FF+FFS</td>
<td>$M_5 \equiv \left[ \frac{k}{T} \frac{1-2 \theta \tilde{H} + \theta^2}{1-\tilde{H}^2} + \frac{T-k}{T} \frac{1-2 \theta H + \theta^2}{1-H^2} \right] + \frac{\sigma_\mu^2}{T \left( 1-H^2 + 1-\tilde{H}^2 \right)} + \frac{\sigma_\epsilon^2}{T \left( 1-H^2 + 1-\tilde{H}^2 \right)}$</td>
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### TABLE 1b

<table>
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<td>$V_1 = \sigma_m^2 / T$</td>
</tr>
<tr>
<td>2) FB</td>
<td>$V_2 = G^2 M_2$</td>
</tr>
<tr>
<td>3) FB+FF</td>
<td>$V_3 = G^2 M_3 + \sigma_m^2 / T$</td>
</tr>
<tr>
<td>4) FB+FFS</td>
<td>$V_4 \equiv \frac{k}{T} \frac{1-2 \theta \tilde{H} + \theta^2}{1-\tilde{H}^2} G^2 + \frac{T-k}{T} \frac{1-2 \theta H + \theta^2}{1-H^2} G^2$ $+ \frac{\sigma_\mu^2}{T \left( 2-G \right)} \left( G \left( 1-\tilde{H}^2 \right) - \tilde{g} \left( 1-\tilde{H}^2 \right) \right)$</td>
</tr>
<tr>
<td>5) FB+FF+FFS</td>
<td>$V_5 \equiv \frac{k}{T} \frac{1-2 \theta \tilde{H} + \theta^2}{1-\tilde{H}^2} G^2 + \frac{T-k}{T} \frac{1-2 \theta H + \theta^2}{1-H^2} G^2$ $+ \frac{\sigma_\mu^2}{T \left( 2-G \right)} \left( G \left( 1-\tilde{H}^2 \right) - \tilde{g} \left( 1-\tilde{H}^2 \right) \right) + \frac{\sigma_m^2}{T}$</td>
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### TABLE 2

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<th>Scheme</th>
<th>output variance</th>
<th>input variance</th>
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### TABLE 3

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<th>Optimal $\tilde{G}$</th>
<th>output variance</th>
<th>input variance</th>
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<td>not used</td>
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### TABLE 4

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<th>Optimal $\tilde{G}$</th>
<th>output variance</th>
<th>input variance</th>
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Table 1. (a) Long-run Mean Squared Errors at the Output for Five Alternative Control Schemes; (b) Corresponding Long-run Adjustment Variances.

Table 2. Relative Performance of the Schemes in Table 1. An example with $T = 100$, $\theta = .8$, $\sigma_\alpha = 1$, $\sigma_\mu = 2$, $\sigma_\varepsilon = .5$, $G = .2$, $\tilde{G} = .8$, and $k = 3$.

Table 3. Relative Performance of the Optimal Unconstrained Schemes of the Types Considered in Section 3. An example with $T = 100$, $\theta = .8$, $\sigma_\alpha = 1$, $\sigma_\mu = 2$, $\sigma_\varepsilon = .5$, and $k = 3$.

Table 4. Relative Performance of the Optimal Constrained Schemes With Input Variance Fixed At (a) 0.06, (b) 0.0525, (c) 0.045. The parameters are $T = 100$, $\theta = .8$, $\sigma_\alpha = 1$, $\sigma_\mu = 2$, $\sigma_\varepsilon = .5$, and $k = 3$. 

page 21