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ABSTRACT

Component tolerances for assembled products are often set with the help of the error transmission formula. However, this approach requires knowledge of the partial derivatives of the functional relationship between the component dimensions and the assembly quality characteristic. In many practical situations, those quantities are not easily obtained. In this article we will demonstrate a how a combined use of computer aided design (CAD) and design of experiments (DOE) can be used to obtain partial derivatives of the functional relationship without knowing an explicit mathematical expression for it. With knowledge of the partial derivatives, we can use the error transmission formula to establish functional tolerances. The intent of the present article is to demonstrate, with some examples, an idea and a set of techniques that can be used to set functional tolerances for mechanical components and assemblies.

Keywords: Specifications, Design of Experiments, Tolerancing Components

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Component tolerances for assembled products are often set with the help of the error transmission formula. However, this approach requires knowledge of the partial derivatives of the functional relationship between the component dimensions and the assembly quality characteristic. In many practical situations, those quantities are not easily obtained. In this article we will demonstrate a how a combined use of computer aided design (CAD) and design of experiments (DOE) can be used to obtain partial derivatives of the functional relationship without knowing an explicit mathematical expression for it. With knowledge of the partial derivatives, we can use the error transmission formula to establish functional tolerances. The intent of the present article is to demonstrate, with some examples, an idea and a set of techniques that can be used to set functional tolerances for mechanical components and assemblies.

Introduction

Mass production and assembly lines, a vital basis for our modern economy and high standard of living, are founded fundamentally on the subtle concept of producing components within narrow tolerance limits. Tolerances allow for the assembly of components into complex finished products that, despite variability in the individual components, produces only an acceptable variation in the final product. In fact, Hounshell (1984) shows that it was largely the inability to manufacture components consistently within narrow limits that held back the widespread use of interchangeable parts and hence mass production from its modern beginning in the late 18th century, to the early 20th century.

Unfortunately, the science, or should we say, the art, of setting realistic tolerances for complex products is today still in an embryonic state. Many manufacturers use rules of thumb that often result in tolerances that are too tight and needlessly expensive or too wide resulting in excessive assembly difficulties or unsatisfactory assembly performance.

A more formal approach is based on theoretical calculations of the assembly's variance based on the contributions coming from the individual components' variances and knowledge of the functional relationship between input and the output using the first order error transmission formula, see e.g. Montgomery (1996). This approach has been used at least since Shewhart (1931, p. 256), and provides the theoretical foundation for much research in statistical tolerancing including ours and many of the recent computer based methods; see Bjørke (1989) for computer based tolerancing and Srinivasan and Voelcker (1993) for a recent state-of-the-art assessment.

An important new development is called functional tolerancing. In this approach the output functionality of a particular system is bracketed with tolerances for how much variability can be allowed without appreciably affecting the ability of the product to function and meet customers needs. From this and knowledge of the transfer function establishing the relationship between the design or input variables and the output quality, it is desired to find a set of tolerances for each of the design or input variables. These concepts are discussed in Srinivasan, et al. (1995), Taylor, et al. (1994) and Zhang, et al. (1997). We will therefore use the term "functional tolerancing" to describe tolerancing based on a model that relate charactersitics of components to the functionality of the assembly in meeting customers' needs.

One problem common to the approaches described above is that they all require the knowledge of a precise mathematical functional relationship between component parameter values and the assembly quality characteristic of interest. They are therefore of limited practical use when such relationships are not available; this is frequently the case because these relationships are either impossible to specify or too cumbersome to establish. Many practical engineers, cumbersome and expensive as it may be, therefore ultimately resort to physical tests to verify the usefulness of a particular set of tolerances.

A new experimental approach for functional tolerancing, useful when no theoretical transfer function is available, has recently been suggested by Bisgaard (1997). With this approach several prototype components are made to exacting specifications either a little below or above the nominal dimensions according to a designed experiment. These perturbed physical prototype components are then assembled either in all possible ways or according to some other experimental design. The quality characteristic of interest is then measured. From the experiments we then get empirical estimates of the partial derivatives; those can in turn be used with the error transmission formula (expression (1) below) to set realistic tolerances.

One disadvantage of this, and for that matter any experimental approach, is that it requires the manufacture and assembly of costly physical prototypes. When the assembly quality characteristic is a geometric dimension, and the problem, one of setting geometric tolerances, we will show that it might not be necessary to make costly physical prototypes. Using instead computer aided design (CAD) software, we might perturb the component drawings using two-level factorial experiments, assemble the parts on the screen to a final assembly, and measure, again on the screen, the geometric influence of these perturbations. The drawing of the perturbed parts is facilitated by editing and modifying a master file containing the nominal design drawing. In the CAD drawing, the geometric deviations of the assembly characteristic from the nominal will be obtained from the CAD drawings and used to estimate the partial derivatives of the design function with respect to each of the design variables. This will give us the information needed to use the error transmission formula to set tolerances.

In this article, we will outline the idea of setting tolerances using CAD and experimental design (DOE). First we will introduce the theoretical foundation for our approach. This is followed by an application to the design of a simple square bracket. For this particular design it is easy to establish the functional relation. We can therefore compare the experimental results with those obtained from theoretical calculations. After this initial example, we move on to demonstrate the application of our approach to a more complex design problem where the theoretical function is harder to establish. In the final section of this article, we allow ourselves to speculate about the future applications of our approach, its possible disadvantages, and what would be needed in terms of computer software to make it useful for the general design engineering public.

Notation and Theoretical Basis

Let $\mathrm{x}'=(x_1,x_2,\dots,x_n)$ be a vector of the n critical design dimensions for a particular assembled product, and $y=f(\mathrm{x})$ a function, often called the design function, relating the individual component design dimensions to the assembly quality characteristic y of interest. In production the dimensions will, of course, vary and thus be random variables $\mathbf{X}'=(X_1,X_2,\dots,X_n)$, where the X_i's are assumed to be independent with $E\{\mathbf{X}\}=\mu$, its nominal and $V\{X_i\}=\sigma_i^2$. Expanding the transfer function $Y=f(\mathrm{X})$ in a first order Taylor series around the nominal dimensions and applying the variance operator, shows that the assembly variance $V\{Y\}=\sigma_Y^2$ is approximately given by

$$\sigma_Y^2 \approx \sum_{i=1}^{n}\left(\left.\frac{\partial f}{\partial x_i}\right|_{\mathbf{x}=\mu}\right)^2 \sigma_i^2 \qquad (1)$$

also known as the *error transmission formula.*

In many applications, it is common to set the assembly tolerance T_Y to be $T_Y=6\times\sigma_Y$. If a similar requirement is imposed on the individual tolerances $T_i, i=1,\dots,n$ such that $T_i=6\times\sigma_i$ then

$$T_Y^2 \approx \sum_{i=1}^{n}\left(\left.\frac{\partial f}{\partial x_i}\right|_{\mathbf{x}=\mathbf{x}_0}\right)^2 T_i^2 \qquad (2)$$

Equations (1) and (2) are the theoretical basis for most statistical tolerancing theory. For an application of (1) to analytic parameter design, robustness and tolerancing, see Bisgaard and Ankenman (1993).

When the design function f is a linear function $y=a_1x_1+\dots+a_nx_n$ with coefficients $a_i=\pm1, i=1,\dots,n$ as would be the case in simple geometric tolerance stack-up, (2) leads to the well-known tolerance formula $T_Y=\sqrt{T_1^2+\dots+T_n^2}$.

The problem of applying (1) and (2) in practical tolerancing work is that the design function $y=f(\mathrm{x})$ is rarely known as a mathematical function except for the classical tolerance stack-up problem, and other examples with simple mathematical structure. We will therefore not easily be able to obtain the partial derivatives $\partial f/\partial x$, and hence apply the above theory.

In some applications, rather than an explicit mathematical function, a computer code might be available describing the functional relationship. For example, the code might be a complex finite element model. In such cases, direct calculus based differentiation is usually not feasible. However, if the design function can be assumed to be reasonably smooth in a small neighborhood around the nominal value, it is possible to obtain estimates of the partial derivatives by numerical differentiation. Specifically, if $f(\mathrm{x})$ is represented by a subroutine of computer code, we might use central differences

$$\frac{\partial f}{\partial x_i} \approx \frac{\Delta f}{\Delta x_i} = \frac{f(x_1,\dots,x_i+h,\dots,x_n)-f(x_1,\dots,x_i-h,\dots,x_n)}{2h} \qquad (3)$$

to obtain an estimate of the partial derivatives. We notice for later reference, that this approach will require $N = 2n$ function evaluations.

As an alternative we might use simple forward differencing

$$\frac{\partial f}{\partial x_i} \approx \frac{\Delta f}{\Delta x_i} = \frac{f(x_1,\ldots,x_i+h,\ldots,x_n) - f(x_1,\ldots,x_n)}{h}. \quad (4)$$

This reduces the number of function evaluations to only $N = n+1$. However, both methods are not very reliable if there are errors, rounding or otherwise, in the calculations. As a rough calculation, treating errors as random, if the variance of an individual calculation is σ^2 the variance of the partial derivatives calculated from (3) will be $\sigma^2/(2h^2)$ and $2\sigma^2/h^2$ using (4). Moreover, if we use (4) the estimates of the derivatives will not be independent since they all rely on the same function evaluation at the base.

A third alternative is to calculate the finite differences from a two-level factorial or fractional factorial experiment usually used for physical experiments (see e.g. Box, Hunter and Hunter, 1979, chapter 10 and 12). Since it can safely be assumed that second and higher order interactions are negligible for this application, the number of function evaluations can be reduced to $N=2^k$ where k is the smallest integer such that $N \geq n+1$.

Using a two-level factorial, the variance of the derivative estimates, using the same assumptions as above, will only be $\sigma^2/(Nh^2)$. Thus, the larger the number of factors, n, and hence the function calls, N, the smaller the variance. This is quite different from the traditional estimates using (3) and (4) where the variance is the same regardless of how many function calls are made. The factorial design method for obtaining numerical derivatives therefore is much to be preferred when there might be measurement errors in the calculations as will be the case for our application below.

One inconvenience of the fractional factorial design approach is that for a large number of individual dimensions, n, we might have to perform a rather large number of function evaluations. For example, if the number of dimensions is $n = 15$ we can use a $2^4 = 16$ run experiment. However, if we increase the number of factors by one so that $n = 16$, then the smallest integer k satisfying the inequality $2^k \geq n+1$ is $k = 5$. Thus we need to perform $2^k = 32$ function evaluations or 15 more than we would need using (4). In other words, the problem is that the differences between two successive powers of two, $2^k - 2^{k-1}$, is 2^{k-1} and thus increases rapidly.

As we already indicated, it is safe to assume for our application, that second order and higher derivatives will be small. Moreover, the error transmission formula (1) only requires knowledge about first order derivatives. We can therefore use Plackett-Burman (PB) designs, see e.g. Kotz and Johnson (1983). It should be noted, however, that these first order designs are not standard fractions of two-level factorials, have complex alias or bias structures, but are nevertheless two-level orthogonal arrays. The advantage of using PB designs is that they come in sizes of $N = 4k$, k=1,2..., and thus increase linearly in size. It is therefore relatively easy to find designs that are close to the minimum of $n+1$ runs. We will demonstrate this in our second example below.

The computer experiment approach discussed above for obtaining numerical derivatives need not be restricted to applications where the design function is specified as a subroutine written in traditional computer code. In fact in the approach described below, it is recognized that a CAD drawing can be considered a mathematical function. That is, besides providing graphical images, CAD can be considered a function that produces numbers, for example, two or three dimensional Cartesian coordinates, distances, and angles. That function can therefore, by appropriate perturbations, be used to provide numerical estimates of partial derivatives of that particular geometric feature of interest and hence estimates of its variance as a function of the component features.

After this brief overview of some of the basic theory and notation needed to explain the our method, we will now turn to a simple example to demonstrate the idea in detail.

Tolerancing the Angle of a Simple Triangular Bracket

In this section we will demonstrate the application of our idea to the triangular bracket shown in Figure 1. It consists of three steel bars with dimensions x_A, x_B and x_C, pinned together with precision ground hardened steel dowel pins. The assembly characteristic of interest is the angle θ be-

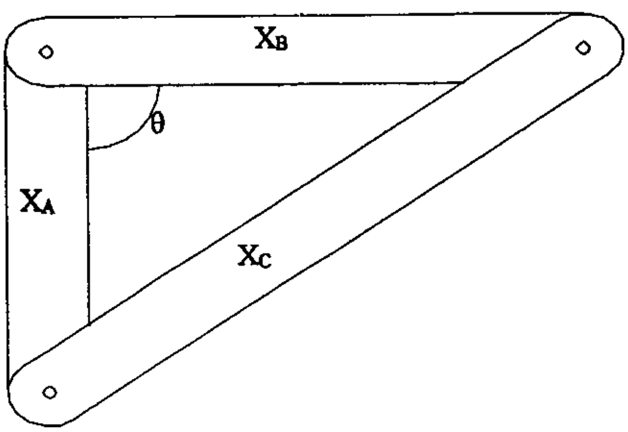

Figure 1. Drawing of the simple square bracket.

Factors	Levels		
	–	0	+
x_A: Length of vertical bar	299.9 mm	300.0 mm	300.1 mm
x_B: Length of horizontal bar	399.9 mm	400.0 mm	400.1 mm
x_C: Length of diagonal bar	499.9 mm	500.0 mm	500.1 mm

Table 1. *The three factors and their levels (perturbations).*

tween components A and B. For simplicity we will assume that other dimensions, for example the pin hole diameters and pin diameters, are manufactured so precisely that we need not worry about their variational influence on θ. Moreover, the nominal value for θ is required to be 90°. The nominal dimensions for x_A, x_B and x_C are therefore chosen to be 300 mm, 400 mm and 500 mm respectively. However, in production, each will deviate slightly from their respective nominal dimensions and have variances σ_A^2, σ_B^2 and σ_C^2.

Let us now assume that we do not know the functional relationship between x_A, x_B and x_C and θ and use this example to show how we can find the variance of θ using CAD and design of experiments. From (1) it can be seen that we need estimates of the partial derivatives. These can, as indicated above, be estimated from a two-level factorial experiment. Since we have three dimensions we will in this case use a full 2^3 factorial design. The factor levels, or perturbations, around the nominal values are given in Table 1 and the design matrix is given in Table 2. The estimated effects computed the usual way and the partial derivatives obtained by dividing the effects by the difference between the factor levels are given in Table 3. (Note that in this and other tables in this article, we will carry along more digits than usual. This is to provide the reader with an impression of the precision of the estimates. Moreover, we have also not used scientific notation to make it easier to compare the order of magnitude of the numbers.)

From this CAD "experiment" we then conclude that the partial derivatives estimates are

$$\frac{\partial f}{\partial x_A} \approx -0.002\,499\,79, \quad (5)$$

$$\frac{\partial f}{\partial x_B} \approx -0.003\,333\,54, \quad (6)$$

and

$$\frac{\partial f}{\partial x_C} \approx 0.004\,166\,49. \quad (7)$$

Incidentally, it is worth noting that the second and third order effects, corresponding to the higher order mixed partial derivatives, are two orders of magnitude smaller than the first order effects, see Table 3. This appears to be quite consistent with theoretical considerations and helps justify the assumptions made earlier that higher order derivatives could be ignored. Indeed, for most manufactured products the transfer function would typically be very nearly linear over the narrow range specified by a tolerance interval.

			Response θ	
x_A	x_B	x_C	Degrees	Radians
–	–	–	90.009 552	1.570 954 7
+	–	–	89.980 910	1.570 454 8
–	+	–	89.971 347	1.570 287 9
+	+	–	89.942 735	1.569 788 5
–	–	+	90.057 327	1.571 788 5
+	–	+	90.028 643	1.571 287 9
–	+	+	90.019 098	1.571 121 3
+	+	+	89.990 454	1.570 621 3

Table 2. *Design matrix and the response in degrees and radians.*

By inserting (5), (6) and (7) into the error transmission formula (1) we have that the variance of θ approximately is

$$\begin{aligned}\sigma_\theta^2 &\approx (-0.002\,499\,79)^2\sigma_A^2 \\ &+(-0.003\,333\,54)^2\sigma_B^2 \\ &+(0.004\,166\,49)^2\sigma_C^2\end{aligned} \quad (8)$$

Expression (8) can be used to set tolerances for the bracket to achieve the tolerance the designer deems necessary. In a typical use of (8), called *tolerance synthesis*, the design engineer inserts values for σ_A^2, σ_B^2 and σ_C^2 and calculates the resulting σ_θ^2. This approach can also be used in spreadsheet calculations to perform if–then type calculations where the component variances are changed to evaluate the consequences of assigning different component variances. Another use of (8) is for *tolerance allocation*. Here a required value for σ_θ^2 is given and the designer then allocates tolerances to the individual dimensions x_A, x_B and x_C and hence specify σ_A^2, σ_B^2 and σ_C^2 often with an eye to manufacturing costs such that the requirement to σ_θ^2 is satisfied.

Factors	Effects (Degrees)	Effects (Rad.)	$\partial f / \partial x$ = Effects/2, rescaled (Rad./mm)
A	−0.028 645	−0.000 499 958	−0.002 499 79
B	−0.038 200	−0.000 666 707	−0.003 333 54
C	0.047 744	0.000 833 299	0.004 166 49
AB	0.000 018	0.000 000 305	0.000 015 3
AC	−0.000 018	−0.000 000 322	−0.000 016 1
BC	−0.000 010	−0.000 000 165 8	−0.000 008 29
ABC	0.000 002	0.000 000 043 6	0.000 021 8

Table 3. Effects in degrees, in radians and the estimated partial derivatives which are the effects divided by the scale factor 0.2, the difference between the high and low levels in Table 1.

For the bracket in Figure 1, it should be obvious that it is not necessary to resort to an experimental approach to get the partial derivatives. Simple trigonometry shows that the functional relationship between θ and x_A, x_B and x_C is given by

$$\theta = f(\mathrm{x}) = \cos^{-1}\left(\frac{x_A^2 + x_B^2 - x_C^2}{2x_A x_B}\right) \tag{9}$$

The partial derivatives of (9) with respect to each of the independent variables with the nominal values for x_A, x_B and x_C inserted are therefore

$$\frac{\partial\theta}{\partial x_A} = -\left(\frac{x_A^2 - x_B^2 + x_C^2}{x_A\sqrt{(2x_A x_B)^2 - (x_A^2 + x_B^2 - x_C^2)^2}}\right) = -0.002\,500\,00, \tag{10}$$

$$\frac{\partial\theta}{\partial x_B} = -\left(\frac{-x_A^2 + x_B^2 + x_C^2}{x_A\sqrt{(2x_A x_B)^2 - (x_A^2 + x_B^2 - x_C^2)^2}}\right) = -0.003\,333\,33, \tag{11}$$

and

$$\frac{\partial\theta}{\partial x_C} = \left(\frac{2x_C}{\sqrt{(2x_A x_B)^2 - (x_A^2 + x_B^2 - x_C^2)^2}}\right) = 0.004\,166\,67. \tag{12}$$

From this, we see that the estimates of the partial derivatives obtained from our approach, (5) to (7), are virtually identical to the correct theoretical values in (10) to (12). Thus we have demonstrated, at least for this example, that partial derivatives obtained by our approach are very good. A single example obviously does not prove anything about how the method will perform in general. But it does, we think, lend credibility to the proposed method for obtaining partial derivatives via CAD and design of experiments.

This example was simple, and it was easy to compare the experimentally estimated derivatives with those obtained analytically. We will now make the example slightly more complicated and apply our technique again. The interesting feature of this next example is that it would be difficult to obtain the partial derivatives by any alternative method we know because of the difficulty of establishing the design function mathematically.

A More Realistic Bracket

In the previous section we somewhat unrealistically assumed that the diameters of the holes for the pins in each end of the three bars were manufactured very precisely. It was therefore not necessary to include their variances in the error transmission calculation. However, as any machinist will testify, drilling holes is a notoriously inaccurate process. Not only may the location of the holes be associated with error, but so are their diameters. Moreover, each of the dowel pin diameters may vary.

While it is feasible to involve this many factors, to do so might obscure what we are trying to demonstrate. In what follows, we will therefore assume that only the diameters of the holes and, as in the previous example, the distances between them vary. Even such a moderate increase in the complexity will, however, make the problem so complicated that it will be quite difficult to establish an analytic expression for the angle θ. Hence, it is not practically feasible to use (1) to assess the variability using the traditional calculus based methods, or any other method based on numerical differentiation or Monte Carlo simulation of a known function.

Adding the diameters of the two holes in each bar to the three length dimensions constitute a total of nine factors as shown in Table 4. The smallest two–level fractional factorial available is a 2^{9-5} design comprising a total of 16 individual "trials." That means 6 "trials" more than the absolute minimum. Since each "trial" involves laborious modifications of a CAD master file this is unfortunate. A more cost effective alternative would be desirable, espe-

Dimension:		Level		
		–	0	+
Length of component A	x_A	299.9 mm	300.0 mm	300.1 mm
Length of component B	x_B	399.9 mm	400.0 mm	400.1 mm
Length of component C	x_C	499.9 mm	500.0 mm	500.1 mm
Diameter of upper hole of A	x_{AU}	12.05 mm	12.15 mm	12.25 mm
Diameter of lower hole of A	x_{AL}	12.05 mm	12.15 mm	12.25 mm
Diameter of left hole of B	x_{BL}	12.05 mm	12.15 mm	12.25 mm
Diameter of right hole of B	x_{BR}	12.05 mm	12.15 mm	12.25 mm
Diameter of upper hole of C	x_{CU}	12.05 mm	12.15 mm	12.25 mm
Diameter of lower hole of C	x_{CL}	12.05 mm	12.15 mm	12.25 mm

***Table 4.** Factors and their levels for the second experiment.*

cially since the previous example gives us reasonable confidence in the numerical estimates of the partial derivatives. We will therefore use a 12 run Plackett–Burman design with one added center point as shown in Table 5. We will not only use this design because it is more cost effective, but because it will provide us an opportunity to make a pitch for this class of experiments, often scorned for general application because of it complex aliasing structure, but extremely useful for tolerancing applications.

Adding the variability of the diameters of the holes complicates the CAD assessment of the response. Since the hole diameters vary, the three bars do not rotate about fixed axes. To cope with this problem, we will assume that the bracket is under the influence of gravity. Moreover, we assume that the bars are perfectly parallel to their centerlines, that they do not twist, and only move in two

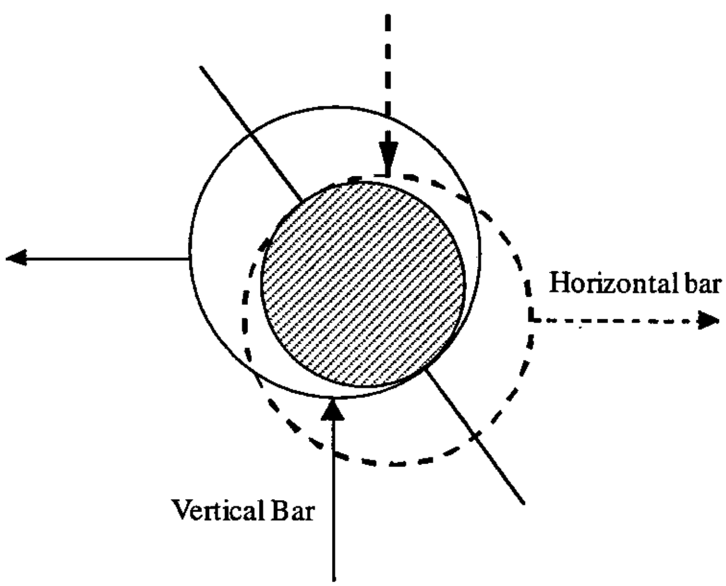

***Figure 2.** The assembly of two bars with oversized holes and a dowel pin. Dimensions are exaggerated for illustration.*

dimensions. With these assumptions, the bars will come to rest such that the forces are in equilibrium and the pins pinched in the holes as in the much exaggerated diagram shown in Figure 2.

To appreciate the problem of measuring the angle it should be noted that the relative location of the pin in Figure 2 can be obtained if θ and the lengths of each component parts are known. With this knowledge it will be possible to make a *free body diagram*, see e.g. Shigley and Mischke (1989). However, the changes in the bars and the holes are so small that the change in θ will also be very small. We will therefore in the free body analysis, when locating the points where the pins and holes touch each other, simply assume that $\theta = 90°$. This approximation does not seem to matter much for this example. We also assume the dowel pins are all 12.00 mm in diameter. With these assumptions the relative location of the pins, and hence the axis about which the bars rotate, can be found from a free body analysis of the whole design. The 13 measurements are shown in the right hand column of Table 5.

On the basis of the thirteen observations shown in Table 5, we can now estimate nine main effects corresponding to the nine partial derivatives as well as two dummy effects. The dummy effects are based on the two linear contrasts corresponding to the two unused columns of the 11 column design matrix. These dummy effects should theoretically be zero except for the influence of error since they are not associated with any effects. They can therefore serve as a check for how well we have executed the experiment and measured the responses. Moreover, they can be used in an analysis of variance as an estimate of the residual error variance. The eleven effects and the corresponding partial derivatives are shown in Table 6, and an Analysis of Variance presented in Table 7.

The analysis of variance displayed in Table 7 shows with an F–ratio of 1654 that the nine partial derivatives are overwhelmingly significant. A breakdown into individual de-

Run	x_A	x_B	x_C	x_{AU}	x_{AL}	x_{BL}	x_{BR}	x_{CU}	x_{CL}	x_{d1}	x_{d2}	Response
1	+	−	+	−	−	−	+	+	+	−	+	89.91481
2	+	+	−	+	−	−	−	+	+	+	−	89.83392
3	−	+	+	−	+	−	−	−	+	+	+	89.93463
4	+	−	+	+	−	+	−	−	−	+	+	89.94416
5	+	+	−	+	+	−	+	−	−	−	+	89.83510
6	+	+	+	−	+	+	−	+	−	−	−	89.88549
7	−	+	+	+	−	+	+	−	+	−	−	89.88699
8	−	−	+	+	+	−	+	+	−	+	−	89.92632
9	−	−	−	+	+	+	−	+	+	−	+	89.85586
10	+	−	−	−	+	+	+	−	+	+	−	89.84790
11	−	+	−	−	−	+	+	+	−	+	+	89.86660
12	−	−	−	−	−	−	−	−	−	−	−	89.96933
13	0	0	0	0	0	0	0	0	0	0	0	89.89304

Table 5. *The Placket Burman design with one center point for nine factors, two dummy factors and the response. The response is in degrees.*

grees of freedom, not provided here, also shows each of the individual effects are overwhelmingly significant.

As a conclusion for this example, we can now show that the angle θ expressed in radians for small changes in the individual dimensions approximately is given by the following linear equation

$$\begin{aligned}\theta \approx\ & 1.56891 - 0.00259(x_A - 300) - 0.00314(x_B - 400) \\ & +0.00413(x_C - 500) - 0.00198(x_{AU} - 12.5) \\ & -0.00190(x_{AD} - 12.5) - 0.00185(x_{BL} - 12.5) \\ & -0.00212(x_{BR} - 12.5) - 0.00197(x_{CU} - 12.5) \\ & -0.00222(x_{CD} - 12.5)\end{aligned} \tag{13}$$

where the constant term is obtained from the designed experiment. Incidentally, it is interesting to note that 90° is approximately 1.570 796 radians. The average angle is therefore slightly biased below 90°, an observation further corroborated by comparison with the center point observation. The design engineer might therefore want to compensate for this bias by changing the length of one or several of the bars. An obvious choice would be to extend the length bar C, which has a positive coefficient, by 0.4570 mm to eliminate the bias.

The variance can likewise be approximated by

$$\begin{aligned}\sigma_\theta^2 \approx\ & (-0.00256)^2\sigma_A^2 + (-0.00314)^2\sigma_B^2 \\ & +(0.00413)^2\sigma_C^2 + (-0.00198)^2\sigma_{AU}^2 \\ & +(-0.00190)^2\sigma_{AD}^2 + (-0.00185)^2\sigma_{BL}^2 \\ & +(-0.00212)^2\sigma_{BR}^2 + (-0.00196)^2\sigma_{CU}^2 \\ & +(-0.00222)^2\sigma_{CD}^2\end{aligned} \tag{14}$$

and with this equation, the design engineer can either assess the variance of the assembly based on given vari-

Factor	Effect (Degrees)	Effect (Rad.)	Effect/2, rescaled (Rad/mm)
Main Effects			
A	−0.029 73	−0.000 518 81	−0.002 594 0
B	−0.035 94	−0.000 627 29	−0.003 136 4
C	−0.047 28	0.000 825 25	0.004 126 2
AU	−0.022 73	−0.000 396 79	−0.001 983 9
AL	−0.021 75	−0.000 379 66	−0.001 898 3
BL	−0.021 19	−0.000 369 76	−0.001 848 8
BR	−0.024 28	−0.000 423 73	−0.002 118 7
CU	−0.022 52	−0.000 393 02	−0.001 965 1
CL	−0.025 48	−0.000 444 73	−0.002 223 6
Dummies			
d1	0.000 993	0.000 017 33	–
d2	0.000 200	0.000 003 48	–

Table 6. *Estimated effects in degrees and radians as well as the partial derivatives in radians per mm.*

ances of the individual component dimensions, the so-called tolerance synthesis, or allocate tolerances to the individual dimension such that a certain required variance of θ can be accommodated. Notice incidentally that the partial derivatives for *A, B* and *C* are very close to those we obtained for the simpler bracket in the first example.

For tolerance allocation, it is interesting to note that the absolute value of the partial derivatives provides an indication of how sensitive or robust the design is to variability in the individual dimensions. For example, we see that the partial derivatives for the length of the three bars are about two times as large as those for the diameters. Thus the influence of the variability of the diameters on the variability of θ are about one quarter of those for the length dimensions. It would therefore be wise to focus manufacturing initiatives to reduce the variance on the length dimensions rather than on the hole diameters. However, such evaluations should also take into account the cost of reducing variability. For example, it might be relatively cheap to ream the holes to make the diameters accurate as compared to locating the distances between the holes more accurately. Thus, it might in that case be wise first to consider reducing the hole variation to see if that would reduce the variance of θ enough to satisfy a particular requirement, before resorting to a more expensive process for locating the holes.

Discussion and Conclusion

The method discussed in this article can in principle, we believe, be used to solve a large number of diverse functional tolerance problems for which no other practical method is currently available. We acknowledge, however, that with standard CAD software packages, the approach is today somewhat cumbersome, a fact that might limit its present practical applicability. Moreover, we have not yet applied the method to 3–dimensional problems. That might pose some new practical problems that we are currently not aware of. We also expect that a number of problems like the free body issue encountered in the second example, will be even more complex to handle.

One limitation we see with our approach is that it may only apply to continuously differentiable functions. A few examples have lately been discussed in the literature where the functionality of interest is not continuously differentiable. This is, for example, the case with a hinge assembly, as described by Altschul and Scholtz (1993), Voelcker (1993), and Parratt (1994).

If we may speculate about possible future use of this approach, we hope that software engineers in the forefront of CAD development will be able to automate and hence ease many of the necessary tasks once the approach has been applied to a sequence of increasingly difficult sample problems. For example, we envision that the matrices for the experimental designs can be accessed from a screen menu. Moreover, subroutine modules for the estimation of effects, and hence partial derivatives, as well as the error transmission formula can be built into the CAD software. Another important improvement would be to have a seamless transfer of data from one module to another. We also think that the perturbations of the component dimensions somehow can be automated.

To sum up, we do not see any obvious limitations to the application of our approach to tolerancing mechanical parts as long as the design function can be assumed to be reasonably continuous. We think that this approach will constitute a much needed practical approach to setting realistic and economic tolerances useful for producing manufactured products.

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Source	*df*	Sum of Squares	Mean Squares	*F*-ratio
Main Effects	9	0.022 787 0	0.002 531 89	1 654.83
Residual Error	3	0.000 004 6	0.000 001 53	
Total	12	0.022 791 6		

Table 7. Analysis of variance table for Placket Burman experiment.

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