

Equivariant Maps for the Symmetric Group

Murad Özaydin

Introduction

For a (nontrivial) finite group G the category of topological spaces with a G -action and G -equivariant maps differs sharply from the nonequivariant category in that there may not be any morphisms X to Y . Constant maps are equivariant only if they take values in Y^G , the points of Y fixed by G , which may be empty. When G is \mathbb{Z}_2 , for instance, the Borsuk–Ulam theorem states that there are no \mathbb{Z}_2 -maps from S^n into S^{n-1} (both with the antipodal action). Below, two different generalizations of this will be given, one for free G -spaces (i.e., only 1 in G fixes any point of the space), G arbitrary, the other for symmetric groups with the target Y nonfree.

The universal G -space E_G is a free, contractible cw complex (this determines it up to homotopy). Its n -skeleton will be denoted by E_G^n , i.e., a finite, free, n -dimensional, $(n-1)$ -connected G -complex (its cells are permuted by G). These properties do not determine E_G^n , even up to Euler characteristic, but there always are G -maps from one model for E_G^n into another because of their universal property: Any free G -cw complex of dimension n or less maps equivariantly into E_G^n . Therefore questions of existence of equivariant maps is independent of the model for E_G^n . Thus it is admissible to occasionally phrase some results as if E_G^n were well defined, as in the following.

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Corollary 3.2. There are no G -maps from E_G^n into E_G^{n-1} .

When G is \mathbb{Z}_2 this becomes the Borsuk–Ulam theorem. If G can act freely on a sphere it is included in [14] and [10]. For instance $E_{\mathbb{Z}_p}^{2n-1}$ can be chosen to be S^{2n-1} . For even n and $G = \mathbb{Z}_p$ this result is contained in [4]. The primary motivation of our note was extending the geometric application in [4]. For this it is necessary to consider $G = S_k$, the symmetric group on k letters. The target space Y is no longer free, it is the unit sphere of m -copies of the regular representation minus the trivial representation of S_k (i.e., the orthogonal complement of the diagonal where S_k permutes the coordinates of \mathbb{R}^k). Our second result is:

Theorem 4.2. There is an S_k -map from $E_{S_k}^{m(k-1)}$ into Y if and only if k is not a prime power.

The dimension is chosen to be $m(k-1)$ not only with the geometric application in mind, but also, below this dimension the existence of equivariant maps are easily established. This result is somewhat different than the former because it involves a space with a nonfree action. One implication (nonexistence when k is a prime power) is established in §4, done via equivariant obstruction theory.

The rest of the paper is organized as follows. In §1 we give some historical background for the geometric problem on coincidences of a map from a simplex into Euclidean space. This goes back to work related to Helly's theorem. In §2 the model for E_G^n (of [4]) when G is S_k is discussed and the equivariant problem is stated. This model as well as the equivariant problem came up surprisingly in the recent solution of two combinatorial problems, on splitting necklaces [1] and on chromatic

number questions [2]. Our theorem also makes a small contribution to the latter (Corollary 3.5). In the final §5 we make some remarks about what is still to be done.

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§1. Historical Background

Our starting point is the following result of Radon.

Theorem 1.1 [16]. Any set of $m + 2$ or more points in \mathbb{R}^m can be partitioned into two parts whose convex hulls intersect.

Proof. Only x_1, \dots, x_{m+2} in \mathbb{R}^m need to be considered. Since $x_i - x_1$ ($i > 1$) are linearly independent, real numbers λ_i , not all zero, can be found such that:

$$\sum_{i=1}^{m+2} \lambda_i = 0 \quad \text{and} \quad \sum_{i=1}^{m+2} \lambda_i x_i = 0.$$

Reordering if necessary, assume that λ_i is nonnegative for $i \leq k$ and negative for $i > k$. Let

$$\lambda = \sum_{i=1}^k \lambda_i = \sum_{i=k+1}^{m+2} -\lambda_i = \sum_{i=k+1}^{m+2} |\lambda_i| \quad (>0).$$

Then the convex hulls of $\{x_1, \dots, x_k\}$ and $\{x_{k+1}, \dots, x_{m+2}\}$ both contain

$$\frac{1}{\lambda} \sum_{i=1}^k \lambda_i x_i = \frac{1}{\lambda} \sum_{i=k+1}^{m+2} |\lambda_i| x_i. \quad \square$$

Radon's purpose was to prove Helly's theorem, which follows immediately ([9] p. 108). A generalization considered by R. Rado and Birch [5] involves finding the least number of points in \mathbb{R}^m that can always be partitioned into k subsets whose convex hulls have a point in common. That $(m+1)(k-1) + 1$ suffices was conjectured by Birch (who proved it for $m = 2$) and settled by Tverberg [17]. An algebraic codimension argument shows that this is sharp. An equivalent formulation is in terms of a linear map from the standard simplex

$$\Delta^n = \{(\lambda_0, \dots, \lambda_n) \in \mathbb{R}^{n+1} : \lambda_i \geq 0, \sum_{i=0}^n \lambda_i = 1\}$$

into \mathbb{R}^m . Tverberg's result now states that if $n \geq (m+1)(k-1)$, then there are k disjoint (closed) faces of Δ^n whose images have a common point. Clearly only $n = (m+1)(k-1)$ needs to be considered. The next step is replacing linearity by continuity.

Conjecture 1.2 [3]. Given any map $f : \Delta^n \rightarrow \mathbb{R}^m$ ($n = (m+1)(k-1)$) there are k (closed) disjoint faces of Δ^n whose images have a point in common.

When $k = 2$ (continuous version of Radon's theorem) this was done by Bajmóczy and Bárány [3] by observing the connection with the Borsuk–Ulam theorem. Let X be the space of pairs (x, y) living in disjoint faces of Δ^n . \mathbb{Z}_2 acts on X by switching the coordinates. Assuming the existence of a map $f : \Delta^n \rightarrow \mathbb{R}^m$ with $f(x) \neq f(y)$ for all (x, y) in X , we define

$$\varphi : X \longrightarrow S^{m-1} \text{ by } \varphi(x,y) = \frac{(f(x) - f(y))}{|f(x) - f(y)|}.$$

This is a \mathbb{Z}_2 -map (with the antipodal action on S^{m-1}). Now exhibiting a \mathbb{Z}_2 -map $\psi : S^m \longrightarrow X$ will prove the conjecture for $k = 2$ by contradiction since $\varphi\psi$ violates the Borsuk–Ulam theorem. Recalling that $n = m + 1$ when $k = 2$, ψ can be explicitly constructed, similar to the proof of Radon's theorem above.

$$\text{Let } S^m = \{(\lambda_0, \dots, \lambda_{m+1}) \in \mathbb{R}^{m+2} : \sum_{i=0}^{m+2} \lambda_i = 0, \sum_{i=0}^{m+2} \lambda_i^2 = 1\}.$$

Also let $\lambda = \sum \lambda_i$ where the sum is over nonnegative coordinates. Now $\psi = (\psi^+, \psi^-)$ can be defined by:

$$\psi_i^+(\lambda_0, \dots, \lambda_{m+1}) = \begin{cases} |\lambda_i|/\lambda & \text{if } \lambda_i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and ψ^- by switching the inequality above. It's not hard to see that $\psi : S^m \longrightarrow X$ is a \mathbb{Z}_2 -map. In fact X is \mathbb{Z}_2 -homeomorphic to S^m and this equivariant formulation is equivalent to the Borsuk–Ulam theorem.

The conjecture for arbitrary k can also be attacked analogously, as described in the next section. When k is an odd prime this was carried out in [4] exploiting equivariance under a cyclic permutation. In §3, using the symmetric group S_k , the conjecture is settled for k a power of a prime. This however is all that can be done with the equivariant approach (§4). The conjecture is open for other k , another easy case ($m = 1$) is shown in §5.

§2. The Equivariant Problem

Now we want to state a problem about the existence of equivariant maps for the symmetric group, a negative answer to which proves the conjecture by contradiction.

This approach is slightly different from [4], but certainly inspired by it.

The simplex Δ^N is the convex hull of the standard basis vectors in \mathbb{R}^{N+1} . The convex hull of any subset is a face, and disjoint subsets give disjoint faces. Let

$$X = \{(x_1, \dots, x_k) : x_i \in \Delta^N \text{ belong to pairwise disjoint faces}\}.$$

A map $f : \Delta^N \rightarrow \mathbb{R}^m$ induces $f_k : X \rightarrow \mathbb{R}^{mk}$, given by

$f_k(x_1, \dots, x_k) = (f(x_1), \dots, f(x_k))$. This is an S_k -map, via permuting the coordinates in

the source and the target. The diagonal $D(\mathbb{R}^m) = \{(y, \dots, y) \in \mathbb{R}^{mk}, y \in \mathbb{R}^m\}$ and its

orthogonal complement are invariant subspaces. Hence orthogonal projection onto the

complement $\{(y_1, \dots, y_k) \in \mathbb{R}^{mk} : \sum y_i = 0\}$ is also equivariant. Assuming the

conjecture to be false for f means that f_k composed with this projection does not

contain the origin in its image. Dividing by the norm we obtain an S_k -map:

$$X \rightarrow Y = \{(y_1, \dots, y_k) \in \mathbb{R}^{mk} : \sum_{i=1}^k y_i = 0, \sum_{i=1}^k |y_i|^2 = 1\}.$$

For any k and m and choosing $N = (k-1)(m+1)$, if we can prove that there are no equivariant maps from X into Y the conjecture is then proved by contradiction.

It is easy to see that Y is the representation sphere mentioned in the introduction. X

is a cw complex of dimension $N - k + 1 = m(k-1)$ whose cells are products of

pairwise disjoint faces of Δ^N freely permuted by S_k . Technically the crucial fact is

([4]) : X is connected up to the top dimension (i.e., any map from the unit sphere of

dimension less than X into X can be extended to the unit ball). Thus X is a model for $E_{S_k}^{N-k+1}$. A cleaner proof (of a generalization) is now available, [2] Lemma 4.4, which uses the homotopy equivalent simplicial complex of the nerve of the covering of X by its maximal cells.

The standard model for E_G^n is the $(n+1)$ -fold join of the discrete set G . This seems very different from the space X above. It is all the more surprising that X has come up in the recent solutions of two different combinatorial problems. In [2] a generalization (also an E_G^n , $G = S_k$) is the simplicial complex associated to a Kneser hypergraph, in [1] it is the space of k -partitions of an N -splitted necklace. Moreover, in both, the results follow from the nonexistence of equivariant maps into Y (for k a prime as in [4]), the same target above!

The existence of an S_k -map $X \rightarrow Y$ does not lead to a negative result, as in all three applications mentioned the equivariant maps constructed have further restrictions imposed by the problem. For both nonexistence (§3) and existence (§4), the following (easy) result is relevant.

Lemma 2.1. A subgroup H of S_k has fixed points in Y iff it is not transitive.

Proof. The point (y_1, \dots, y_k) in Y is fixed by H iff $y_i = y_j$ when i and j are in the same H -orbit. If H is transitive then $\sum y_i = 0$ implies all $y_i = 0$, a contradiction. If there is more than one H -orbit, it is not hard to construct H -fixed points in Y , starting with any nonzero point in \mathbb{R}^m . □

§3. Nonexistence of Equivariant Maps

We give a few generalizations of the Borsuk–Ulam theorem and obtain partial solutions to Conjecture 1.2 and a question raised in [2] involving the chromatic number of k -regular hypergraphs. In the first G is any nontrivial finite group and E_G^n is an n -dimensional, $(n-1)$ -connected finite free G -complex ($n \geq 1$).

Proposition 3.1. Let X be a free G -complex of dimension at most n . If $f : E_G^n \rightarrow X$ is a G -map then the induced homomorphism $f_* : H_n(E_G^n; \mathbb{Q}) \rightarrow H_n(X; \mathbb{Q})$ is nonzero.

Proof. By the universal property of E_G^n , there is a G -map $h : X \rightarrow E_G^n$. The composition hf is a self G -map of the finite free G -complex E_G^n , so its Lefschetz number is divisible by $|G| > 1$ ([12],[15]). But E_G^n has nontrivial homology only in dimensions 0 and n . The contribution of $H_0(E_G^n; \mathbb{Q})$ is 1 and $(hf)_* = h_*f_*$. Hence f_* can not be trivial. \square

Corollary 3.2. There are no G -maps from E_G^n into E_G^{n-1} .

Proof. E_G^{n-1} is $(n-1)$ -dimensional, so $H_n(E_G^{n-1}; \mathbb{Q})$ is trivial. \square

When E_G^n is a sphere, the approach of [10] uses a similar divisibility with coincidence numbers instead of Lefschetz numbers. This approach to Borsuk–Ulam type results goes back to G. Hirsch, a modern reference is [12]. There are divisibility

results for Lefschetz numbers of equivariant self maps for nonfree actions [15]. These don't help to rule out G -maps from E_G^n into a nonfree G -space Y because there are no G -maps from Y into E_G^n . The next result where the target is not free uses the idea of cohomological index [11]. The proposition above can also be proved using this method, but we consider the given argument more elementary. Below $H^*(-; \mathbb{F}_p)$ denotes reduced cohomology with coefficients the finite field \mathbb{F}_p .

Lemma 3.3. Let G be an elementary abelian p -group. Assume that X and Y are G -complexes satisfying:

- a) $H^i(X; \mathbb{F}_p) = 0$ if $i \leq d$ for some integer d ;
- b) Y is a finite complex without any G -fixed points, the only nonzero reduced homology of Y is $H^d(Y; \mathbb{F}_p)$ on which G acts trivially. Then there is no G -map from X into Y .

Proof. If $F : X \rightarrow Y$ is a G -map, then we would have a commutative triangle

$$\begin{array}{ccc} (X \times E_G)/G & \longrightarrow & (Y \times E_G)/G \\ & \searrow & \swarrow \\ & E_G/G & \end{array}$$

where E_G is a contractible free G -complex and G acts diagonally on the products. The cohomology of E/G is the cohomology of the group G . At the cohomology level we have:

$$\begin{array}{ccc}
 H^{d+1}((X \times E_G)/G; \mathbb{F}_p) & \longleftarrow & H^{d+1}((Y \times E_G)/G; \mathbb{F}_p) \\
 & \swarrow & \searrow \\
 & H^{d+1}(G; \mathbb{F}_p) &
 \end{array}$$

The Serre spectral sequence of the fibration $X \longrightarrow (X \times E_G)/G \longrightarrow E_G/G$ at the E_2 level (see [7] Ch II) has $E_2^{ij} = H^i(G; H^j(X; \mathbb{F}_p))$, which is zero for $0 < i \leq d$. Hence $H^{d+1}(G; \mathbb{F}_p) \longrightarrow H^{d+1}((X \times E_G)/G; \mathbb{F}_p)$ is injective. Commutativity of the diagram above implies that the other map (when X is replaced by Y) is also injective. Then the spectral sequence for Y would collapse ([7] 14.1) contradicting a result of Borel ([6] p 164) because Y does not have G -fixed points. \square

We want to apply this lemma to our set-up with $X = E_{S_k}^{m(k-1)}$, Y the representation sphere of §2 ($d = m(k-1) - 1$), and $k = p^r$, a power of a prime. The regular action of an elementary abelian p -group of rank r on itself embeds it in S_k as a transitive subgroup. Then by Lemma 2.1 Y has no fixed points. Lemma 3.3 is now applicable, all the other hypotheses are easy to check, and implies that equivariant maps do not exist even for this subgroup.

Corollary 3.4. There are no S_k -maps from $E_{S_k}^{m(k-1)}$ into Y when k is a prime power. \square

In [4] nonexistence of equivariant maps is established for a cyclic group when k is a prime. This also follows from Proposition 3.1 as the action on the target is then free. As explained in §2 we now have a partial solution to Conjecture 1.2 (when k is a prime power) extending the main result of [4].

Similarly, Corollary 3.4 implies a generalization of [2], 2.1 from k a prime to a prime power. The question is the chromatic number of k -uniform hypergraphs. A

simplicial complex $C(H)$ is assigned to a hypergraph H on which S_k acts freely. A coloring of H with $m + 1$ colors induces an equivariant map from $C(H)$ into $(\Delta^m)^k \setminus D(\Delta^m)$, an invariant subspace of $\mathbb{R}^{mk} \setminus D(\mathbb{R}^m)$. This maps equivariantly to our sphere Y (§2). If $C(H)$ is $(m(k-1) - 1)$ -connected then $E_{S^k}^{m(k-1)}$ can map equivariantly into $C(H)$ (see §4). The composition from $E_{S^k}^{m(k-1)}$ to Y would contradict Corollary 3.4 when k is a prime power. We have just proved:

Corollary 3.5. For any k -uniform hypergraph H , where k is a prime power, if $C(H)$ is $(m(k-1) - 1)$ -connected then H is not $(m+1)$ -colorable. \square

§4. Obstructions and Equivariant Maps

In the previous section we showed that there are no equivariant maps from $X(E_{S^k}^{m(k-1)})$ into Y when k is a prime power. Now we want to demonstrate that k being a prime power is also necessary for the nonexistence of equivariant maps. In [4] and [2] only the equivariance under the (cyclic) subgroup permuting the k indices cyclically is considered. In general any model for E_G^n is also a E_H^n for any subgroup H (dimension and connectivity are independent of G and the restriction of a free action is still free). Our methods also yield that if we restrict to this cyclic subgroup of the symmetric group, there are no equivariant maps if and only if k is a prime. This provides a converse to the result of [BSS].

Equivariant maps will be constructed via obstruction theory ([18] Ch V §5, [8] Ch II). Our strategy is to reduce the problem to Sylow p -subgroups. Below, G is a

finite group as usual, G_p denotes a Sylow p -subgroup for each prime p dividing the order of G , and d is a positive integer.

Lemma 4.1. Let X be a $(d+1)$ -dimensional free G -complex and let Y be a $(d-1)$ -connected G -complex (if $d = 1$ then also assume that $\pi_1(Y)$ is abelian). There is a G -map from X into Y if and only if there are G_p -maps from X into Y for each p .

Proof. One direction is trivial because a G -map is a G_p -map for all p . Conversely, assume that there is a G_p -map f_p for each p . A standard way of constructing a map from a complex is to do it skeleton by skeleton. At each stage the map is defined on the boundary of a cell and the problem is extending it to the interior. This can always be started at the 0-skeleton (discrete set of points) by choosing arbitrary values (as long as the target Y is nonempty). When X is a free G -complex, to insure that we have a G -map at each stage, the extension problem is considered for a single cell in each G -orbit (equivariance determines it on the rest of the orbit). If the map f is already defined (equivariantly) on the j th skeleton X^j , then for a representative $(j+1)$ -cell the problem is extending the composition of the attaching map with f from S^j to the disk D^{j+1} . When Y is $(d-1)$ -connected as assumed, there is no problem extending f up to X^d . For each $(d+1)$ -cell in X we have a map $S^d \rightarrow Y$. Only the homotopy class of this map is relevant for purposes of extending to D^{d+1} . A basis for cellular $(d+1)$ -chains of X is given by $(d+1)$ -cells, so we have just defined the "obstruction cocycle" $o(f) \in C^{d+1}(X; H_d(Y))$. ($H_d(Y)$ is isomorphic to $\pi_d(Y)$ by our hypothesis on Y). This is always a cocycle (in our case this is immediate because X is $(d+1)$ -dimensional). Moreover, since f on X is equivariant, $o(f)$ is an equivariant cocycle (regarding $H_d(Y)$ as a

G -module). The map can be extended to X^{d+1} if and only if $o(f)$ is zero. If $o(f)$ is an equivariant coboundary then f can be modified (equivariantly), only on d -cells, to obtain another map whose obstruction cocycle is zero, which can then be extended equivariantly to $X^{d+1} = X$. Thus all we need to show is that $o(f)$ is an equivariant coboundary.

For a subgroup K of G , let $H_K^{d+1}(X; H_d(Y))$ be the quotient of K -equivariant cocycles by K -equivariant boundaries. Any G -equivariant cocycle is also K -equivariant, so we can consider the restriction maps:

$$H_G^{d+1}(X; H_d(Y)) \longrightarrow H_K^{d+1}(X; H_d(Y))$$

Since $f : X^d \rightarrow Y$ is a K -map, the obstruction to extending it K -equivariantly (after a possible modification on X^d) is given by the cohomology class $[o(f)]$ in $H_K^{d+1}(X; H_d(Y))$. Now let $K = G_p$, and consider the G_p -map f_p restricted to X^d . Y is $(d-1)$ -connected, so the restriction of f and f_p are G_p -homotopic on X^{d-1} . Then $[o(f)] = [o(f_p)] = 0$, because f_p can be extended. Now we know that the restriction of $[o(f)]$ to any G_p is zero. But there is also a transfer map

$$H_K^{d+1}(X; H_d(Y)) \longrightarrow H_G^{d+1}(X; H_d(Y))$$

\rightarrow induced from the cochain map sending α to $\sum_{Kt} \alpha t$ where t runs over a left transversal for K in G . The composition of restriction with the transfer is just multiplication by the index $[G:K]$. Therefore $[G:G_p] [o(f)] = 0$ for all p dividing $|G|$, hence $[o(f)] = 0$. \square

The obstruction theory in the proof of Lemma 4.1 is almost non-equivariant because the source is a free G -complex X . The more general treatment of [8] Ch II is not directly applicable though, the target Y does not have a G -fixed basepoint. However the hypothesis on Y enables us to pass from $\pi_d(Y)$ to $H_d(Y)$ via the Hurewicz isomorphism, and the latter has a well defined G -action so it provides the coefficients of the equivariant cohomology groups $H_K^*(X; H_d(Y))$.

Theorem 4.2. There is an S_k -map (resp. \mathbb{Z}_k -map, where \mathbb{Z}_k is generated by the cyclic permutation $(1\ 2\ \dots\ k)$) from $E_{S_k}^{m(k-1)}$ (resp. $E_{\mathbb{Z}_k}^{m(k-1)}$) into Y if and only if k is not a prime power (resp. prime).

Proof. One direction is Corollary 3.4 (or 3.1 in the cyclic case). For the other note that $E_{S_k}^{m(k-1)}$ and Y satisfy the hypothesis of Lemma 4.1 with $d = m(k-1) - 1$. Hence the question reduces to the case of Sylow p -subgroups (for each p). When k is not a prime power (or not a prime, in the cyclic case), the p -group G_p can not act transitively on the k indices. Thus Y has a G_p -fixed point by Lemma 2.1. Then (constant) G_p -maps exist. \square

§5. Concluding Remarks

While nonexistence of equivariant maps for k a prime power (Corollary 3.4) imply Conjecture 1.2, and gives a lower bound for the chromatic number of k -uniform hypergraphs (Corollary 3.5), existence of equivariant maps for the remaining k (Theorem 4.2) don't lead to counterexamples. In both cases the equivariant map constructed to get a contradiction has further restrictions that we have not been able to

exploit yet. A simple direct argument for Conjecture 2.1 can be given for $m = 1$
 (courtesy of R. Zivaljevic).

Proposition 5.1. Given any map $f : \Delta^{2(k-1)} \rightarrow \mathbb{R}$ there are k pairwise disjoint faces of $\Delta^{2(k-1)}$ whose images have a common point.

Proof. There are $2k-1$ vertices of $\Delta^{2(k-1)}$, whose images in \mathbb{R} are totally ordered. Consider the edges whose endpoints have the largest and smallest values, the second largest and the second smallest, ..., etc. The k th face is the vertex whose image has the middle value. By the intermediate value theorem this value is also in the image of all the other $(k-1)$ faces. \square

Even after Theorem 4.2 there is hope of getting some bounds using equivariant topology. When there are G -maps from E_G^n into Y the next question is: How about from E_G^{n+1} into Y ? Raising the dimension of E_G^n corresponds to raising the dimension of the simplex in Conjecture 1.2 or the connectivity of $C(H)$ in Corollary 3.5. There is some evidence [13] that equivariant K -theory may detect nonexistence of equivariant maps from E_G^N into Y for large enough N . In this case G should be taken to be a transitive cyclic subgroup since K_G is determined by the restrictions to the cyclic subgroups, and up to conjugacy there is a unique transitive one. This N may be very large of course, but the construction in Theorem 4.2 barely works and N could well be $n + 1$.

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