1. Introduction

Time scale calculus analyzes dynamic systems on time scales which generalize difference and differential equations. Results from this area of focus allow for the unification of continuous and discrete cases. A time scale, denoted \( \mathbb{T} \), is a non-empty closed subset of \( \mathbb{R} \). A time scale interval for \( a \in \mathbb{T} \), where \( a < \xi \), is defined to be \( [a, \xi] \cup \{ \xi \} \).

Let \( \mathbb{T} \) be a time scale.
- The forward jump operator, \( \sigma: \mathbb{T} \to \mathbb{T} \), is defined by \( \sigma(t) = \sup \{ s \in \mathbb{T} : s < t \} \).
- The backward jump operator, \( \rho: \mathbb{T} \to \mathbb{T} \), is defined by \( \rho(t) = \inf \{ s \in \mathbb{T} : s > t \} \).
- The graininess operator, \( \mu: \mathbb{T} \to [0, \infty) \), is defined by \( \mu(t) = \sigma(t) - t \).

Classification of points on \( \mathbb{T} \):
- \( a \) is said to be right-dense if \( \lim_{t \to a} t = a \).
- \( a \) is said to be left-dense if \( \lim_{t \to a} t = a \).
- \( a \) is said to be right-scattered if \( \lim_{t \to a} t = \rho(a) \).
- \( a \) is said to be left-scattered if \( \lim_{t \to a} t = \sigma(a) \).
- \( a \) is said to be isolated if \( \mu(a) = 0 \).

Lemma 1. We say that a function \( f: \mathbb{T} \to \mathbb{R} \) is \( \Delta \)-differentiable for some right-dense point \( t \in \mathbb{T} \) provided that \( f^\Delta = \lim_{h \to 0} \frac{f(t + h) - f(t)}{h} \) exists.

Lemma 2. Suppose \( f: \mathbb{T} \to \mathbb{R} \) and \( t \in \mathbb{T} \). If \( \Delta \)-differentiable and \( f \) is continuous at \( t \), then the \( \Delta \)-derivative is \( f^\Delta(t) = \lim_{h \to 0} f(t + h) - f(t) \).

Definition 3. Another operator that can be used on time scales is the \( \Delta \)-integral defined to be \( \int_a^b f(t) \Delta t = \mu(b) \int_a^b f(t) \) if \( F: \mathbb{T} \to \mathbb{R} \) is an anti-derivative of \( f: \mathbb{T} \to \mathbb{R} \) provided \( F^\Delta = f \).

A central focus of this project is the solution’s interval of existence. If the solution obtained has an asymptote, we proceed by removing an interval around it, thus creating a new time scale. Our initial value problem is set up as follows:

\[
\begin{align*}
y'(t) &= f(t, y(t)) \\
y(a) &= \alpha \quad \text{where } a \geq 0.
\end{align*}
\]

(1)

Notation: \( \mathbb{T}^+ = [0, \infty) \)

Notation: \( \mathbb{T}_k = [\beta_k, \alpha_k] \cup [\beta_{k+1}, \alpha_{k+1}] \cup \cdots \cup [\beta_{2k-1}, \alpha_{2k-1}] \cup [\beta_{2k}, \alpha_{2k}] \), where \( \beta_k = \frac{k}{2} + \frac{1}{4k^2} + \frac{1}{4} + k \) and \( \alpha_k = \frac{k}{2} + \frac{1}{4k^2} + \frac{1}{4} + k \) for \( k \geq 2 \).

2. Motivation

Since the time scale looks like \( \mathbb{R} \) near 0, we are able to treat (1) as a differential equation, \( y' = f \). By using separation of variables, we obtain the solution described below. The interval of existence is \( (-\infty, \frac{1}{2}) \). This particular graph to the right shows the case for when \( n = 1 \), which is clearly represented by the asymptote at \( t = 1 \). Since \( \lim_{t \to 0} y(t) = 0 \), and \( y(t) \) is continuous for \( t < 0 \), it follows that the interval of existence is \( (-\infty, \frac{1}{2}) \).

3. Interval of Existence

**Theorem 4.** On the time scale \( \mathbb{T}_n = [0, \beta_n] \cup [\alpha_n, \beta_n] \cup [\alpha_n, \beta_n] \cup \cdots \cup [\alpha_{n-1}, \beta_{n-1}] \cup [\alpha_{n}, \infty) \), the solution of \( y'' = y^2, y(0) = a \) is

\[
y(t) = \frac{1}{t - a}.
\]

We solve the differential equation and the solution has an asymptote when \( t = \frac{1}{2} \).

### Upper Bound

Finding the upper bound of the interval of existence can be done with two steps.

1. First is to find the total length of the gaps between the intervals:

\[
s_n = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}.
\]

2. Second is to find the total length of the intervals, which can be done using

\[
\sum_{k=1}^{\infty} 2\epsilon_k \leq 2\epsilon_1.
\]

### Facts

- \( S_n \) is monotone increasing.
- \( S_n \leq \frac{1}{n} \forall n \in \mathbb{N} \).

Therefore, by the Monotone Convergence Theorem:

\[
\sum_{k=1}^{\infty} |\beta_k - \alpha_k| \leq \frac{1}{n}.
\]

References


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