1. What is an $A_{\infty}$ Algebra?

A multilinear map $\alpha \in C^n(V) = \text{Hom}(V^n, V)$ extends to a coderivation of $T(V)$ by

$$\alpha(v_1 \cdots v_n) = \sum_{i=0}^{n} (-1)^{i}v_1 \cdots \hat{v}_i \cdots v_n\alpha(v_1, \ldots, \hat{v}_i, \ldots, v_n).$$

Using the formula above we are able to identify the Lie algebra $\text{Coder}(T(V))$ of coderivations with

$$C(V) = \text{Hom}(T(V), V) = \bigoplus_{n \geq 0} C^n(V).$$

As a consequence, any coderivation $d$ has a decomposition as a power series $d = \sum_{n \geq 0} d^n$. An odd coderivation $d = d_1 + d_2 + \cdots$ such that $[d, d] = 0$ is called an $A_{\infty}$ algebra structure on $V$.

An $A_{\infty}$ structure $d$ determines a cohomology operator $D$ on $\text{Coder}(T(V))$ by $D(\phi) = d_1 \phi$ for $\phi \in C(V)$. Then $D^2 = 0$, so we can define the cohomology $H(d)$ by

$$H(D) = \ker(D)/\text{im}(D).$$

When $d$ consists of a single term $d_n$ we can redefine this cohomology to define $H^n(d)$ for all natural numbers $n$.

2. Extensions of a Vector Space

What is an Extension?

An extension of an algebra $W$ by an algebra $M$ is represented by a short exact sequence

$$0 \to M \to V \to W \to 0,$$

where $V$ is the vector space $V = M \oplus W$, equipped with some algebra structure.

In the language of algebra, we have $M$ as an ideal in $V$, and $W = V/M$ is the quotient algebra. Why we are interested in this construction is that we want to determine the moduli space of all algebras on $V$ by looking at the moduli spaces of algebras on smaller dimensional spaces.

If $V$ has a proper nontrivial ideal $M$, then we can use the idea of extensions to express $V$ as an extension of the algebra $W = V/M$ by $M$. Then, we also have to understand the case when $V$ has no proper nontrivial ideals. In this case we say $V$ is simple.

When $V$ is $\mathbb{Z}_2$-graded we require that ideals be graded, meaning it is a graded subspace of a vector space $V$.

3. The moduli space of $A_{\infty}$ algebras

An invertible even linear map $\lambda : V \to V$ extends in a natural way to a coalgebra automorphism of $T(V)$. Moreover, if $a, b \in C^n(V)$ is even for $k > 1$, then $\exp(a) = \lambda^n\exp(b)$ is always defined. An arbitrary coalgebra morphism $y$ can be written in the form $y = \exp(a_{n+2}x_{n+1}y_{n+2})\cdots$, where $\lambda \in \text{GL}(V)$ and $a_2 \in C^2(V)$. Then

$$y^n = (\prod_{k=1}^{n} \exp(-ad_{n-k})(\lambda^k)).$$

The important fact about the above formula is that it is computable.

We say that $d$ and $d'$ are equivalent $A_{\infty}$ algebra structures if there is a coalgebra automorphism $\lambda$ of the tensor coalgebra such that $\psi^*(\lambda) = d'$ and write $d \sim d'$.

Theorem 1 Suppose that

$$d = d_0 + d_1 + \cdots; \quad d' = d_0 + d_1 + \cdots.$$

Then $k = l$ and there is a linear automorphism $\lambda$ of $T(V)$ such that $\psi^*(\lambda) = d'$. Because of this theorem, we know the first step in classifying the $A_{\infty}$ algebras is to classify all nonisomorphic $A_{\infty}$ algebras consisting of a single term $d_0 \in C^0$.

4. Three Types of deformation Problems

We give three types of deformation problems which have the same formal structure:

1. Formal deformations of an algebra. Let $d = d(\varphi)$, where $\varphi \in \text{Hom}(T(V), V)$.

2. Extension of a degree $n$ coalgebra to a unital $A_{\infty}$ algebra. Let $d = d(\varphi)$, where $\varphi \in \text{Hom}(T(V), V)$ for $n \geq 2$.

3. Extending a degree $n$ codifferential $d$ on $T(V)$ to a degree $n$ codifferential $\rho$ on $T(M)$, $n$ codifferential on $V = \mathbb{Z} \otimes M$.

5. A Descending Sequence of Cohomology

The set of equations $\xi_0 = 0$ gives a sequence of cohomology operators $D_k$, each defined on the previous cohomology space $H_{k-1}$. Define

$$D_k(\varphi) = \{\varphi + \kappa\}, \quad H_k = \ker(D_k)/\text{im}(D_k).$$

Since $D_2^2 = 0$, $H_2$ is well defined, and since

$$D_k(\varphi, \chi) = [D_k(\varphi), \chi] + (-1)^{K} (\varphi, D_k \chi),$$

$H_0$ is a graded Lie algebra. We prove the following theorem:

**Theorem 2** An element $\varphi$ gives rise to a cohomology class $[\varphi] \in H_0$ precisely when there are elements $\varphi_1, \ldots, \varphi_n$ such that the sequence of equations

$$[\varphi_1, \varphi] = 0, \quad [\varphi_i, \varphi_{i+1}, \varphi] = 0, \quad \cdots$$

where $\xi_0 = [\varphi_1, \varphi] = [\varphi_1, \varphi_2, \varphi] = \cdots = [\varphi_n, \varphi] = 0$.

Then the map $D_{n+1} : H_n \to H_{n+1}$ given by $D_{n+1}([\varphi_1, \varphi_2, \varphi_3]) = [\varphi_{n+2}, \varphi_{n+1}, \varphi]$, $\varphi \in \text{ker}(D_{n+1})$ is well defined, satisfies $D_{n+1}^2 = 0$, and the cohomology $H_{n+1}$ is the $D_{n+1}/\text{im}(D_{n+1})$ has the structure of a graded Lie algebra. Moreover $[\varphi]$, is well defined for all $n$.

The theorem above plays a role in the calculation of extensions of $A_{\infty}$ algebras. In the next slide, we give an example of this construction.

6. $A_{\infty}$ Algebras of degree 3 on a 1|1 dimensional space

<table>
<thead>
<tr>
<th>Codifferential</th>
<th>$H^0$</th>
<th>$H^1$</th>
<th>$H^2$</th>
<th>$H^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_0 = \varphi_1^2$</td>
<td>$d_1 = \varphi_1^2 + \varphi_1^2$</td>
<td>$d_2 = \varphi_1^3 + \varphi_1^2$</td>
<td>$d_3 = \varphi_1^4 + \varphi_1^3$</td>
<td></td>
</tr>
</tbody>
</table>

The codifferential $d_0$ of $d_1 = d_2 = \cdots$ arise as extensions. We have jump deformations $d_2 \to d_3 \to \cdots$, $d_2 \oplus d_1 \to d_3$.

References


Acknowledgments

- Office of Research and Sponsored Programs, UW-Eau Claire
- Department of Mathematics, UW-Eau Claire
- This document was typeset using MTHX