Using Puzzles to Teach Deductive Reasoning and Proof In High School Geometry

by

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A Master's Project

Submitted to the Graduate Faculty in Partial Fulfillment

of

the Requirements for the Degree of

MASTER OF SCIENCE IN EDUCATION

MATHEMATICS

Plan B Curriculum Project

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UNIVERSITY OF WISCONSIN _ RIVER FALLS

June 29, 2012

Abstract

This study examined the effects of incorporating logic puzzles into a high school geometry curriculum to teach deductive reasoning in preparation for instruction on constructing formal proof. Two high school geometry classes were used in the study. One class completed seven days of instruction and practice solving logic puzzles before they learned how to construct formal proofs. The other class only received the traditional instruction. I predicted that the class that was exposed to the logic puzzles would score higher on a deductive reasoning posttest and the unit exam that included constructing formal proofs. Although this group did have a higher mean score on both tests, there was not enough statistical evidence to verify the hypothesis. However, the participants in the class that used puzzles showed significantly greater confidence in constructing proofs and felt less stressed during the unit than the other group, suggesting positive outcomes with the incorporation of puzzles.

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1. Introduction

As a high school geometry teacher, the most challenging part of the curriculum for both students and me has been the construction of formal proof. Students have found it to be unnatural and very difficult to master. It was difficult to explain to them how to justify their thoughts. Frustration always appeared in abundance in my classroom during the weeks in which proving theorems was mandated.

As enrichment material I have always provided a variety of puzzles for my students to solve when they have completed their regular assignments. Many of the students were eager to complete these puzzles and become very efficient in doing so. Some puzzles were more popular than others and there seems to be varying opinions on which are the most difficult. A few students chose not to attempt any of the puzzles.

One semester I noticed that several students, who were having a very difficult time constructing formal proofs, were excelling at solving Battleships puzzles. I began to wonder if there was a way to harness their abilities in using deductive reasoning in the puzzle scenario and apply it to the task of constructing formal proofs. I pondered whether these puzzles had even more educational value than I realized.

2. Literature Review

For many years there has been debate within the mathematics education community regarding the effectiveness of teaching proof in the high school classroom. According to Steen (1999), "Proof is central to mathematical reasoning, yet there is precious little agreement on how, when, why, or to whom to teach it. Its suitability for school mathematics has always been open to question, both on the grounds of pedagogy and relevance" (p. 274).

In 1989 the National Council of Teacher of Mathematics (NCTM) published the document Curriculum and Evaluation Standards for School Mathematics in which it chose to de-emphasize formal proof and instead focused on reasoning. The mathematical reasoning standard stated:

> "Students can be introduced to the forms of deductive argument by examining everyday situations in which such forms arise naturally. … Students can begin to appreciate the power of deductive reasoning by providing simple valid arguments as justification for their solutions to specific problems and for algorithms constructed for various purposes" (p. 144)

The consensus was that it was sufficient for students explain their reasoning process without having to construct a formal written proof.

In 2000, the NCTM published an updated version of their standards which presented a change in this philosophy. It indicated that there needed to be an increased expectation for high school students to construct proofs. A summary of the reasoning standard included:

> "In high school, students should be expected to construct relatively complex chains of reasoning and provide mathematical reasons. . . . They should be able to present

> mathematical arguments in written forms that would be acceptable to professional mathematicians" (p. 58).

Similar statements are found throughout standards that have been developed by individual state departments of education. Teachers assume responsibility for these standards when they accept employment within a state.

Along with consensus from most teachers that proof is a worthwhile component of secondary mathematics curriculum, it is commonly agreed that it is one of the most difficult topics to teach students. The language of formal logic appears as the first obstacle for students. In his 1988 keynote address, Allen describes how students feel frustrated and sometimes even traumatized by the sudden expectation to develop mathematical arguments before knowing the required language needed to make them understandable. This idea was iterated in the 1999 NCTM Yearbook article, "The vocabulary of mathematical truth, rigor, and certainty is not a natural habitat for most students; their world is more empirical, relying on modeling, interpretation, applications" (Steen, 1999, p. 274).

The advanced thought processes required to construct proofs presents another hardship for students. Some believe that constructing formal arguments is too complex for high school students. The solution to helping students develop the skills required for forming a good proof cannot necessarily by found in providing them with instruction on logic. There have even been studies that indicate that an explicit unit on logic was ineffective in improving the ability of high school geometry students to construct proofs (Epp, 2003).

Perhaps the key to success lies in worrying less about teaching the formal vocabulary and rules for writing rigid proofs and more about helping students develop the ability to form reasonable arguments. In 2002, Reiss suggested that "the teaching and learning of proof should not be restricted to presenting a correct proof. It is more important to stress the process of proving rather than to give the outcome of this process" (p. 98).

This focus on explaining why things are true rather than creating a formal proof is a much more important skill for high school students. The role of proof in the classroom is to promote understanding. Hersh emphasized this idea as, "the student needs proofs to explain, to give insight why a theorem is true. Not proof in the sense of formal logic" (1997, p. 162).

So it seems the current belief is that proof is an essential part of secondary mathematics curriculum, especially in its focus to help students learn how to explain mathematical concepts. "Nevertheless, of all the roles of proof, its role in promoting understanding is, perhaps, the most significant from an educational perspective" (Knuth, 2000, p. 3). It is important to not let the difficulty of teaching the syntax of proofs distract from the reason to teach them. For students to be able to communicate their understanding to others is an ultimate goal in teaching them how to construct formal proofs. Steen claimed:

> "The important question about proof may not be whether it is crucial to understanding the nature of mathematics as a deductive, logical science, but whether it helps students and teachers communicate mathematically. Is, perhaps, proof in the school classroom more appropriate as a means than as an end? " (1999, p. 5)

Reiss (2004) made similar comments:

> "The teaching and learning of proof should not be restricted to presenting a correct proof. It is more important to stress the process of proving rather than to give the outcome of this process" (p. 98).

Although the standards indicate that students should eventually be able construct a formal mathematical proof, appropriate instruction should lead them to this skill gradually. It is unreasonable to expect students to write two column or paragraph proofs before they learn to justify single statements.

At the root of all of this discussion remains the fact that a core part of secondary mathematics curriculum includes helping students to become good problem solvers who can explain and justify their solutions with clarity. It is likely that this can be done through means more enticing to students than instructing them on how to write a correct formal proof.

Engaging students can be one of the most difficult tasks required of an effective mathematics teacher. A motivated student responds much more receptively to learning new concepts. According to Lombard, "Because puzzles are fun and challenging, they can teach your students to enjoy and recognize the value of the methods used in problem solving" (2003, p. 3).

Teaching students the importance of persistence and self-confidence is another positive outcome that comes from using puzzles and games in the classroom. Too often students give up in their attempts to find solutions to problems, especially those that are identified as more difficult, e. g. proofs. Moursund wrote:

> "Many puzzles require a concentrated and persistent effort. The puzzle solver is driven by intrinsic motivation and develops confidence in his or her abilities to face and solve challenging problems. Improving persistence and self-sufficiency are important educational goals" (2007, p. 56).

Lombard (2003) agreed:

> "Students are empowered when solving puzzles because they realize they have a chance to do something really cool, and there is a tremendous amount of satisfaction felt upon completion of the task. … Perseverance is taught and cultivated this way" (p. 4)

Some of the most important skills to have when learning how to develop mathematical proof include motivation, perseverance, and self confidence. Students with these assets are more likely to be successful in their arguments.

There exist many more connections between solving puzzles and the improved ability to write proofs. Logical thinking is essential in order to be able to establish a coherent mathematical argument. According to Moursund (2007), puzzle solving requires the use of logical thinking and often requires strategic and creative thinking as well. And even more importantly he stated, “Especially with some mentoring help, students can transfer their increasing puzzle-based logic and problem solving to other situations” (p. 55-56).

A core part of creating proof lies in the ability to articulate the reasoning that leads to conclusions. Explaining one’s approach and justification for the steps in solving a puzzle can be essential to developing the skills required to develop mathematical arguments. This thinking about thinking provides a key component to students developing habits of deductive reasoning (Lombard, 2003). He also wrote:

> “Mathematical games can foster mathematical communication as students explain and justify their moves to one another. In addition, games can motivate students and engage them in thinking about and applying concepts and skills…” (p. 3).

As with any skill, mathematical or otherwise, the ability to explain the process to someone else only improves one’s own understanding.

In order to have the best effect, puzzles and games must be intentionally and thoughtfully included in the curriculum. Especially in the development of the skills necessary for creating proofs, extra thought and reflection on the puzzles must be included. In his guide, Moursund stated:

> "Puzzles are inherently educational. However, some puzzles have much more educational value than others. In addition, the educational value of puzzles can be substantially increased by appropriate teaching and mentoring. Thus, a teacher who is interested in puzzles should have no difficulty justifying the routine integration of puzzles into the curriculum" (2007, p. 74).

As with any teaching tool, using puzzles in the classroom requires careful planning and implementation in order for it to have the desired effect.

As part of this incorporation of puzzles, their link to mathematical proof must be explained explicitly. Mitchell described one way to make this relationship clear:

> "… let us take a quick look at how deductive thought works. In essence this type of thinking allows us to start off with a few statements that we accept as true (imagine being a detective here starting with a few pieces of evidence) and then to apply those statements and the rules of logic to establish the truth of other, new statements. Just as a detective may use a few facts combined with impeccable logic to conclude something new, mathematicians are constantly creating new truths" (2007, p. 2).

If a teacher can help the student make a clear distinction of each logical step in the solving of a puzzle, this method can be paralleled to developing a mathematical proof.

This idea of using puzzles to teach reasoning was the inspiration for my research. It was clear that teaching students how to construct formal proofs is a difficult, but necessary part of the curriculum. Amending the traditional forms of instruction is required. It was also apparent that there are many advantages to using puzzles in the mathematics classroom. I committed to performing a study in my one classroom.

3. Study

A. Research Question

I proposed the question: "Can the use of puzzles in the classroom improve students' deductive reasoning skills and ability to construct mathematical proofs?"

B. Participants

The study was conducted during the fall semester of 2011 at Amery High School. The participants were enrolled in a geometry course that was taught in 90 minute class periods over an 18 week semester. At the time of the study, the students were approximately midway through the course and had completed units on basic vocabulary, properties of lines and angles, and parts and properties of triangles. The next unit in the regular curriculum to be studied focused on using the congruence shortcuts, i.e. SSS, SAS, ASA, and SAA, to determine whether or not given triangles were congruent and included an assessment on their ability to construct a formal proof.

Two geometry classes, one with 27 students and one with 26 students, were observed. One received instruction and practice using several logic puzzles and the other did not.

C. Materials

All subjects were given a pretest (Appendix A) as an informal measure of their deductive reasoning skills. This was a test that I constructed to expose students to a variety of problems requiring the application of logical thinking.

Language independent logic puzzles were acquired, with permission, from the web site www.conceptispuzzle.com to be used with one of the groups. The six different types of puzzles used were Sudoku, Battleships, Hashi, PicAPix, Hitori, and Nurikabe.

During the regular curriculum unit both groups completed homework assignments that required them to apply the triangle congruence theorems. They were also given assignments on which they had to construct flow chart style proofs.

At the end of the unit, the students completed a unit examination (Appendix C), a deductive reasoning posttest (Appendix B) and an informal survey (Appendix E).

D. Procedure

Before beginning the unit, students were given a pretest (Appendix A) on deductive reasoning skills. This test also required students to complete two formal flow chart proofs. The expectation was that most students would find these proofs difficult to complete before the unit.

Both groups of students then completed an investigation using Geometer's Sketchpad® during which they discovered and discussed the triangle congruence theorems (Appendix D). During the next two class periods they practiced using these theorems to determine when triangles were congruent. Some of the work was done as class activities and some was in the form of individual assignments.

At this point each class received differing instruction. One class was exposed to traditional discussion and instruction about solving proofs. We spent three days discussing and practicing them before they were given the unit examination (Appendix C). Following the unit test they were also given a deductive reasoning posttest (Appendix B) and a survey about their experience and comfort level (Appendix E).

The puzzles used were chosen for their language independence and simple rules. It was important for the focus of instruction to be on students' ability to justify their solutions. Although the puzzles are considered to be of equal difficulty, I ordered them according to how I thought students would approach them, from the comfortable to the more challenging. The puzzles varied enough to allow students to practice their deductive reasoning without being limited to any one type of puzzle restriction.

Seven days were spent studying and discussing different types of logic puzzles. Each of the first six class periods was spent studying a different type of logic puzzle. This included discussing the rules, completing a puzzle as a class, completing a puzzle with a partner, and completing a puzzle individually. Besides solving the puzzles, students were asked to document each step of the solution in order to help them learn to verbalize their justifications for each step.

The first puzzle studied was Sudoku. This type of puzzle was chosen because it was familiar to all of the students and most of them had at least attempted to solve Sudoku puzzles in the past. The lesson on the first day of the puzzle unit began with a discussion of the rules of Sudoku. Each student received a copy of the instructions (Appendix F) and we read through it together. Then an image of a puzzle was projected in the front of the classroom. Students took turns identifying where a number could be secured and stated why the placement was justified. This sequence of solving was recorded (Appendix G).

The students then separated into groups of two or three where they solved another Sudoku puzzle (Appendix H) and recorded the sequence of their solution. While completing this they explained their steps to each other to check that all conclusions being made were valid. After the completion of the partner activity, students were given an assignment to solve another Sudoku puzzle, recording the sequence of their solution individually (Appendix I).

This three step procedure allowed students to gradually move from an understanding of the puzzle type to formally justifying each step of a solution. In the large group setting, I tried to direct the discussion so that as many students as possible could practice verbalizing their justification. This allowed for additional discussion about the difference in strong and weak arguments. It also allowed for me to make sure students had a clear understanding of what was required in their justifications before they began their individual practice.

The second day we concentrated on a puzzle called Battleships. This type of puzzle had been offered as an enrichment exercise earlier in the semester, so some students were already familiar with the rules. We followed the same sequence; first discussing the rules of the puzzle (Appendix J), solving and documenting a solution as a class (Appendix K), solving a puzzle in pairs (Appendix L), and finally as an individual assignment (Appendix M). Many students indicated that they preferred the Battleships puzzles to Sudoku.

The third day we looked at Hashi puzzles (Appendixes N-Q). The remaining days we studied PicAPix, Hitori and Nurikabe puzzles (Appendixes R-CC) all in the same manner. Students found that some of the puzzles were easier to document steps for than others. A majority of the students felt that the Nurikabe puzzles were the most difficult to solve.

On day seven, students worked in groups of 2 or 3 on their favorite types of puzzles. They did not have to document steps, but they were asked to verbally justify each step with each other. I provided two practice puzzles of each type (Appendixes DD) and asked each student to complete at least one puzzle of two different types.

Following this seven day puzzle unit, the second class was exposed to the traditional curriculum. As with the first class, we spent three days discussing and practicing formal proofs. During discussions we were able to relate the steps of proof arguments to the steps of solving logic puzzles. Then the class was given the unit test (Appendix C) and the deductive reasoning posttest (Appendix B).

E. Results

The deductive reasoning pretest (Appendix A) and posttest (Appendix B) were both scored on a 50 point scale. The results for each class are detailed in Table 1. The unit test (Appendix C) was scored on a 60 point scale. Table 2 provides a summary of the scores.

Students were also given a survey (Appendix E) on which they rated their own knowledge regarding the triangle congruence theorems and their ability to construct proofs. It also included a question that asked them to indicate their stress level while they were being instructed on how to construct proofs. The mean response for each question is listed in Table 3.

In addition to these formal assessments, I made many observations during the course of the study. It seemed that some of the puzzles had more effect on the students' ability to verbally defend an argument than others. Some of the puzzles had inherent roadblocks to this same skill.

The choice to use the familiar Sudoku puzzle to begin the unit presented a few unexpected issues. One was that many students had already formed opinions about this type of puzzle. Some thought that they were easy while others were convinced that they were too difficult for them to solve. The students who found them to be easy tried to use the phrase "it has to go here" as justification rather than thoroughly explaining their deductive reasoning.

By far, the most instructionally effective puzzle type seemed to be Battleships. The rules were simple, which allowed the students to concentrate on justifying their answers. It was stressed that students should not make a "move" unless they used deductive reasoning to verify that it was a certainty. Verbalizing their progress seemed natural to most of them.

Students were the least patient with the PicAPix and Nurikabe puzzles. Both types require the solver to revisit areas of the puzzle in order complete the solution. Information that is not helpful at the beginning becomes more important as the solver progresses. Many of the students became frustrated with the repetition required to complete each puzzle. Although these later puzzles seemed more difficult for the students, by this time most were able to articulate very specific reasoning for each step to their solution. I believe this was simply because they were more practiced at it.

4. Conclusion

In comparing the results of the pretest, the classes were fairly equal. The class that would not receive instruction and practice with logic puzzles had a lower mean score than the class that would study puzzles, but it was very slight at 0.17 points. The posttest results showed a greater difference with the group that studied puzzles having a mean score 3.31 points higher than the group that did not. However, this was not enough to make a statistically relevant claim.

Similar results were found in the unit test scores. The class that spent seven days working with logic puzzles had a mean score that was 1.39 points higher than the class that did not. A two sample t-test showed that the difference was not great enough to make a statistically valid claim about it.

The analysis of the responses to the survey indicated that there were two questions that received statistically different replies from the two groups. The first of these was related to the amount of stress that students felt while learning how to construct formal proofs. They were asked the question, "How would you rate your stress level while we were studying how to construct proofs?" Each student was to reply with a number from 0 to 10, 0 indicated no feelings of stress and 10 representing extreme feelings of stress. The class that incorporated puzzles before formal proofs had a mean response of 3.22 and the class that did not include puzzles had a mean response of 5.92. A two sample t-test produced calculations that indicated a 99.2% confidence interval that a member of the first group would reply with a lower number than the second group. This illustrates that the group that used puzzles to study deductive reasoning before learning about formal proofs definitely felt less stress. This certainly matches my observations in the classroom. Not only did the group that had used puzzles seem less stressed, they were more willing to try to construct proofs on their own.

Another question on the survey was, “How would your rate your ability to construct proofs now?” A response of 0 indicated no knowledge and a response of 10 would represent a feeling of complete knowledge. The class that used puzzles had a mean response of 7.52 while the other class had a mean of 6.04. The two sample t-test resulted in a confidence interval of 99.9% that a member of the first group would respond higher than a member of the second group. This coincides with not only their willingness to attempt to construct formal proofs, but with the confidence they had about the quality of their proofs. The group that had used logic puzzles to demonstrate deductive reasoning skills were more comfortable in defending statements as we held class discussions about proofs. Although the results of the unit test did not indicate with certainty that the class that used puzzles were better at constructing proofs, the fact that they felt more confident was definitely a worthwhile result.

My continued observations of the classes after the formal study verified that there were advantages to incorporating a study of logic puzzles into the curriculum. Students with this experience were more critical of theorems that we discussed and used. On occasion a student would ask, “Can that be proven?” This resulted in the class constructing a proof of the theorem together, mostly with verbal justifications for each step that I would record on the board. I believe that this resulted in a deeper understanding of many of the proofs.

So regardless of whether or not the incorporation of logic puzzles actually improved the students’ deductive reasoning skills, there were definite positive effects. Students were more confident and felt less stressed. They were more eager to discuss and prove theorems and this resulted in a more thorough understanding of the material. Many of the students even gained a new interest in solving logic puzzles as a pastime. I concluded that incorporating logic puzzles into the geometry curriculum would have positive results.

5. Future Research

The completion of this research project brought with it several ideas that warrant further investigation. One of these included the use of variations of the puzzles to include those with multiple solutions or no solution. Others involved more critical examination of students' confidence levels in regards to completing proofs.

In this project, all of the puzzles were solvable and had a unique solution. This was intentional in order to make a correlation between solving the puzzles step by step to completing formal proofs in the same manner. However, students' growth in ability to form logical arguments could be enhanced by using variations of these puzzles. A puzzle that has multiple solutions could be used in an exercise where the student has to argue why more than one solution is possible. Similarly, a puzzle without a solution could be used to force students to make arguments to prove its lack of solution. This would require the use of different vocabulary, e.g. words like cannot instead of must, in their arguments.

The confidence levels reported by students on their surveys inspired several additional research questions. One of these questions was how students' confidence in their ability to construct proofs relates to their confidence that specific proofs are well written. A logical follow-up to this is the question of how students' confidence level correlates to their actual ability to complete proofs. Also worthy of examination would be comparing the data related to these questions by gender.

It is my intention to make these investigations in the future. I have been convinced that there are positive outcomes to using puzzles in my classroom and will continue to do so. Additional research related to this will allow me to make even more informed decisions to continue to improve my curriculum.

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Table 1

Pretest and Posttest Results

	Pretest Mean Score	Posttest Mean Score	Change In Mean Score	Pretest Median Score	Posttest Median Score	Change In Median Score
Class that did not use puzzles	27.27	37.50	10.23	26	40	14
Class that used puzzles	27.44	40.81	13.37	26	42	16

Table 2

Unit Test Results

	Unit Test Mean Score	Unit Test Median Score	Standard Deviation
Class that did not use puzzles	48.98	50	7.43
Class that used puzzles	50.37	50	5.44

Table 3

Mean Response (from scale of 0-10) of Survey Questions

	Class that did not use puzzles	**Class that used puzzles**
Previous Knowledge of Theorems	0.77	0.74
Present Knowledge Of Theorems	6.96	7.04
Previous Ability to Construct Proofs	.19	.15
Present Ability to Construct Proofs	6.04	7.52
Stress Level While Learning To Construct Proofs	5.92	3.22

Appendix A

Deductive Reasoning Pretest

Using the facts below, indicate whether each of the statements is definitely true (T), definitely false (F) or cannot be determined (CBD).

FACT: All musical instrument players are in the school band.

FACT: All of the trumpet players are sophomores.

FACT: None of the flute players are sophomores.

FACT: Jake plays the tuba.

FACT: Lisa plays the trumpet.

FACT: Kayla is a sophomore.

________ Lisa is a sophomore.

________ Jake is a sophomore.

________ Kayla plays the flute.

________ Kayla plays the trumpet.

________ Kayla plays the tuba.

________ There are more flute players than trombone players.

________ Kayla and Lisa are in the same grade.

________ Kayla, Lisa and Jake cannot all be in the same grade.

Given the information below, list all relationships possible between Amy and Dani.

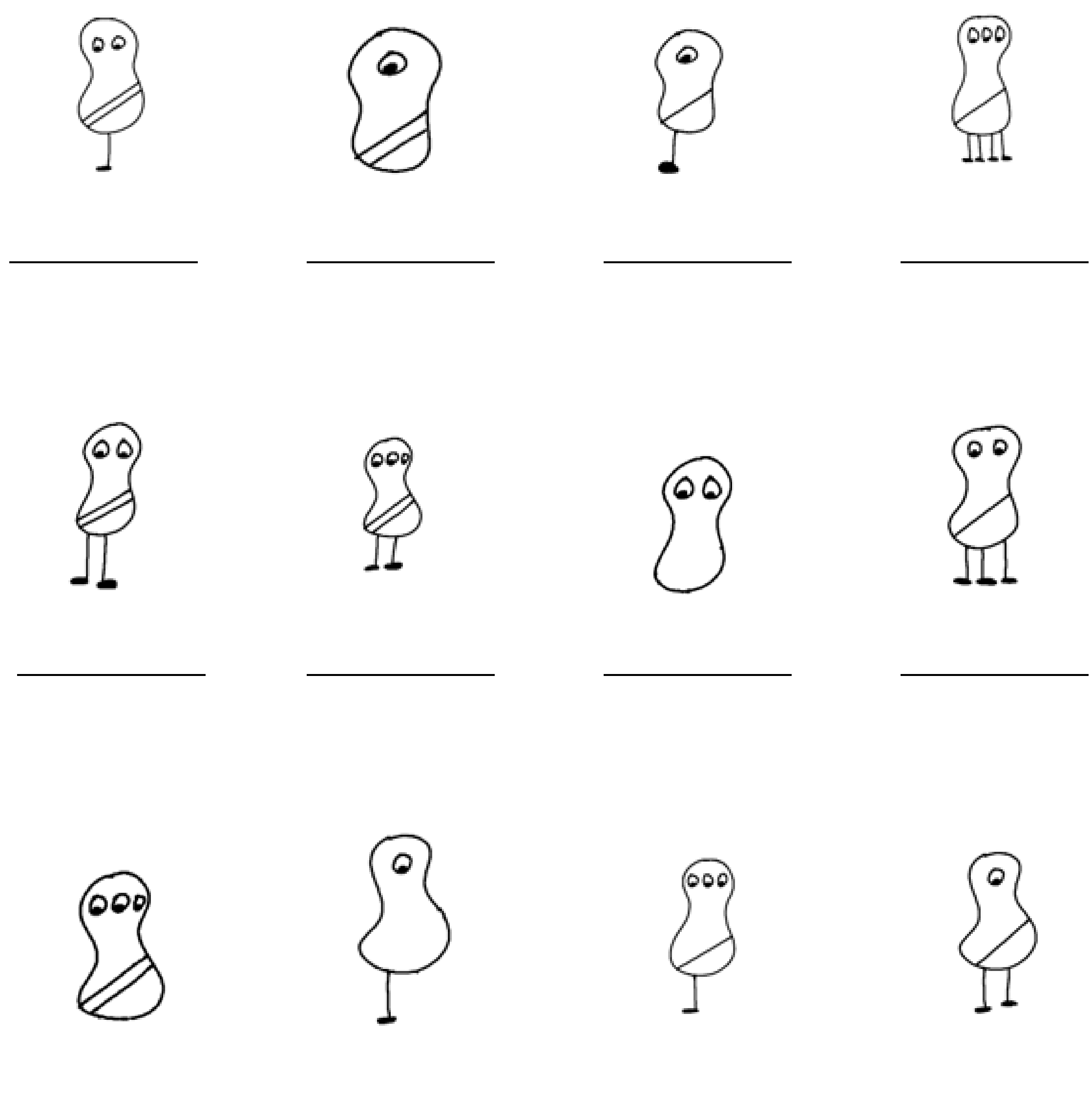
An aunt is: 1) the sister of a parent or 2) the wife of the brother of a parent.

Bailey is Amy’s aunt. Claire is Bailey’s sister. Dani is Claire’s daughter.

Under each picture, write the classification that best represents it.

On the planet Lars, there are many types of living creatures. The Lartian scientists have classified them as follows:

CYCLOIDS: creatures with one eye
PEGOIDS: creatures with one leg
MAXOIDS: creatures that are both cycloids and pegoids
PEGNONS: creatures with no legs
NORMALS: creatures with more than one eye and more than one leg.
ODDBALL: any creature that is not one of the above



When given two statements, determine which conclusions can be made.
Circle the letter or letters of each correct conclusion.

Given: Squares are rectangles.

Rectangles are quadrilaterals.

Conclusions: A) Quadrilaterals are squares.

B) Squares are quadrilaterals.

C) Rectangles are squares.

Given: If a triangle is isosceles, it has two congruent angles.

Triangle XYZ is isosceles.

Conclusions: A) Angles X and Y are congruent

B) Triangle XYZ is acute.

C) Both A and B are true.

D) None of the above.

Number each statement so that the scenario follows a logical order.

_____ Chan, the cat, ran to slurp the spilled milk.

_____ Dan's sneeze caused Jan to jump.

_____ Ann put on some perfume.

_____ When Jan jumped, she spilled her milk.

_____ Since Dan is allergic to perfume, he started sneezing.

_____ The intersection of the perpendicular bisectors is the circumcenter, so X is the circumcenter of the triangle.

_____ Since AX = BX, then $\overline{AX} \cong \overline{BX}$

_____ X is the point of intersection of the perpendicular bisectors of ΔABC

_____ Since the circumcenter is equidistant to the vertices of a triangle, AX = BX

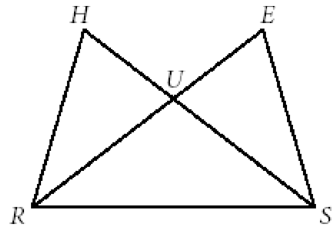
Under each boxed statement of the proof, write why you can make the conclusion that it is a true statement.

<u>PROOF A</u>

Given: $\angle H \cong \angle E$

$\overline{UR} \cong \overline{US}$

Show: $\overline{HR} \cong \overline{SE}$



$\angle H \cong \angle E$	______________
$\overline{UR} \cong \overline{US}$	______________
$\angle HUR \cong \angle EUS$	______________

→ $\Delta RHU \cong \Delta SEU$ ______________

→ $\overline{HR} \cong \overline{SE}$ — CPCT

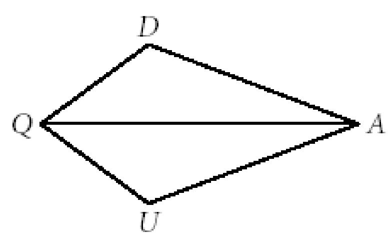
In each box, make sure that there is a true statement that follows a logical argument. Under each boxed statement of the proof, write why you can make the conclusion that it is a true statement.

<u>**PROOF B**</u>

Given: $\angle UQA \cong \angle DQA$

$\angle DAQ \cong \angle UAQ$

Show: $\angle U \cong \angle D$



$\angle UQA \cong \angle DQA$	______________
$\angle DAQ \cong \angle UAQ$	______________
	______________

→ Δ____ $\cong \Delta$ ______ ______________

→ [ ] ______________

Appendix B

Deductive Reasoning Posttest

Using the facts below, indicate whether each of the statements is definitely true (T), definitely false (F) or cannot be determined (CBD).

FACT: All student drivers have a parking permit.

FACT: All of the students who drive trucks have permits for Lot A.

FACT: None of the motorcycle drivers can park in lot A.

FACT: Sarah drives a compact car.

FACT: Kyle drives a truck.

FACT: Ben has a permit for Lot A.

________ Kyle is a parks in Lot A.

________ Sarah is a parks in Lot A.

________ Ben drives a motorcycle.

________ Ben drives a compact car.

________ Ben drives a truck.

________ There are more truck drivers than motorcycle drivers.

________ Ben and Kyle drive the same type of vehicle.

________ Ben, Kyle and Sarah cannot all drive the same type of vehicle.

Given the information below, list all relationships possible between Jessie and Karen.

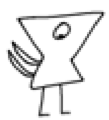
A niece is: 1) the daughter of a sibling or 2) the daughter of your spouse's sibling.

Jessie is Karen's niece. Lori is Karen's sister. Karen is Mandy.s daughter.

Under each picture, write the classification that best represents it.

On the planet Lenus, there are many types of living creatures. The Lenutian scientists have classified them as follows:

TRILOIDS: creatures with three eyes
WAGOIDS: creatures with one tail
MAXOIDS: creatures that are both triloids and wagoids
WAGNONS: creatures with no tail
NORMALS: creatures with less than three eyes and more than one tail
ODDBALL: any creature that is not one of the above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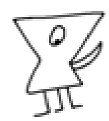
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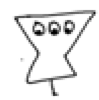
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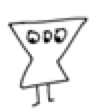
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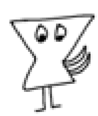
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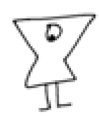
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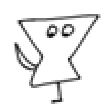
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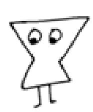
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____


____

When given two statements, determine which conclusions can be made.
Circle the letter or letters of each correct conclusion.

Given: A rhombus is a parallelogram.
Parallelograms are quadrilaterals.

Conclusions: A) Quadrilaterals are parallelograms.
B) A rhombus is a quadrilaterals.
C) A quadrilateral is a rhombus.

Given: If a triangle is isosceles, it has two congruent angles.
Triangle XYZ is isosceles.

Conclusions: A) Angles X and Y are congruent
B) Triangle XYZ is acute.
C) Both A and B are true.
D) None of the above.

Number each statement so that the scenario follows a logical order.

_____ Mae slipped on the banana peel and fell.
_____ Ray went to the store and bought bananas for his family
_____ Jose has to take Mae to the hospital.
_____ When Mae falls she breaks her leg.
_____ Jay ate one of the bananas and dropped the peel on the floor.

_____ The intersection of the perpendicular bisectors is the circumcenter, so X is the circumcenter of the triangle.
_____ Since AX = BX, then $\overline{AX} \cong \overline{BX}$
_____ X is the point of intersection of the perpendicular bisectors of ΔABC
_____ Since the circumcenter is equidistant to the vertices of a triangle, AX = BX

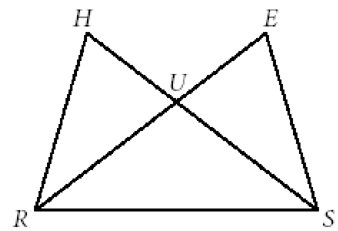
Under each boxed statement of the proof, write why you can make the conclusion that it is a true statement.

PROOF A

Given: $\angle H \cong \angle E$

$\overline{UR} \cong \overline{US}$

Show: $\overline{HR} \cong \overline{SE}$



$\angle H \cong \angle E$

____

$\overline{UR} \cong \overline{US}$

____

$\angle HUR \cong \angle EUS$

____

→ $\Delta RHU \cong \Delta SEU$

____

→ $\overline{HR} \cong \overline{SE}$

CPCT

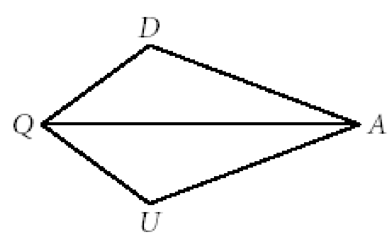
In each box, make sure that there is a true statement that follows a logical argument. Under each boxed statement of the proof, write why you can make the conclusion that it is a true statement.

PROOF B

Given: $\angle UQA \cong \angle DQA$

$\angle DAQ \cong \angle UAQ$

Show: $\angle U \cong \angle D$



$\angle UQA \cong \angle DQA$

____

$\angle DAQ \cong \angle UAQ$

____

____

____

→ Δ____ ≅ Δ ______

____

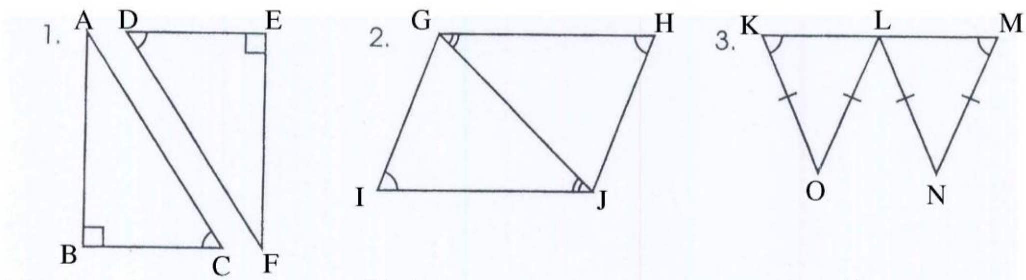
→ ________________

____

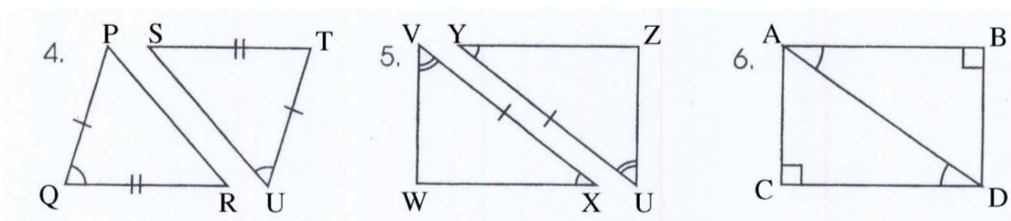
Appendix C

TEST – Triangle Congruence and Proofs

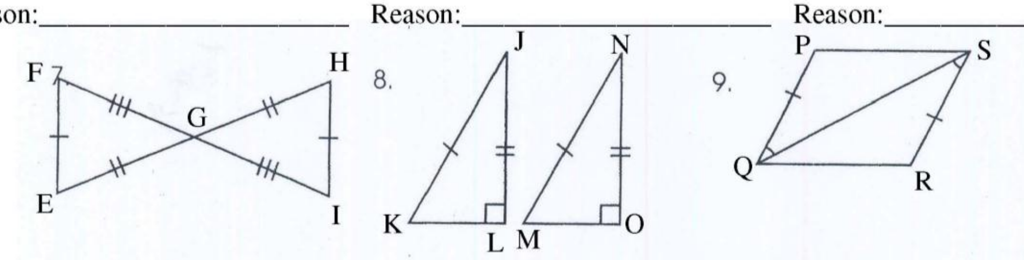
For each problem indicate whether or not there is sufficient information to be certain that the triangles are congruent. Then write the congruence statement and indicate the theorem illustrated.



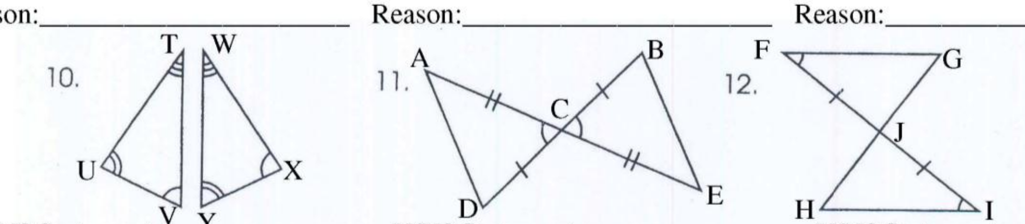
Y/N? Statement:_______________ Reason:______________________
Y/N? Statement:_______________ Reason:______________________
Y/N? Statement:_______________ Reason:______________________



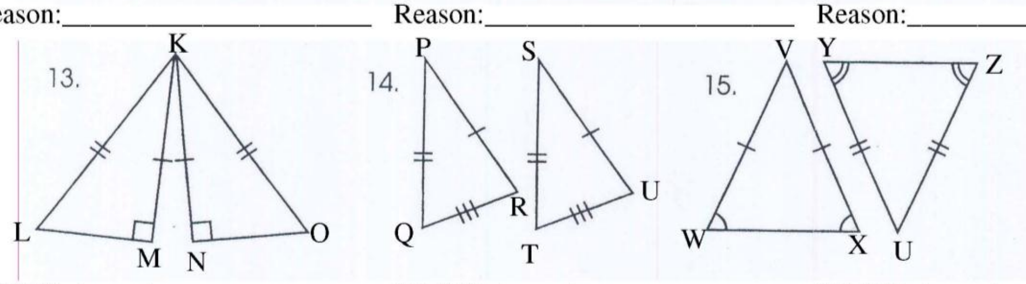
Y/N? Statement:_______________ Reason:______________________
Y/N? Statement:_______________ Reason:______________________
Y/N? Statement:_______________ Reason:______________________



Y/N? Statement:_______________ Reason:______________________
Y/N? Statement:_______________ Reason:______________________
Y/N? Statement:_______________ Reason:______________________



Y/N? Statement:_______________ Reason:______________________
Y/N? Statement:_______________ Reason:______________________
Y/N? Statement:_______________ Reason:______________________



Y/N? Statement:_______________ Reason:______________________
Y/N? Statement:_______________ Reason:______________________
Y/N? Statement:_______________ Reason:______________________

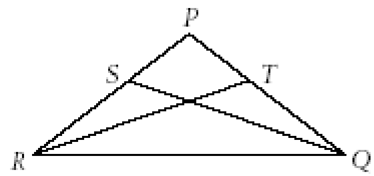
PROOF A

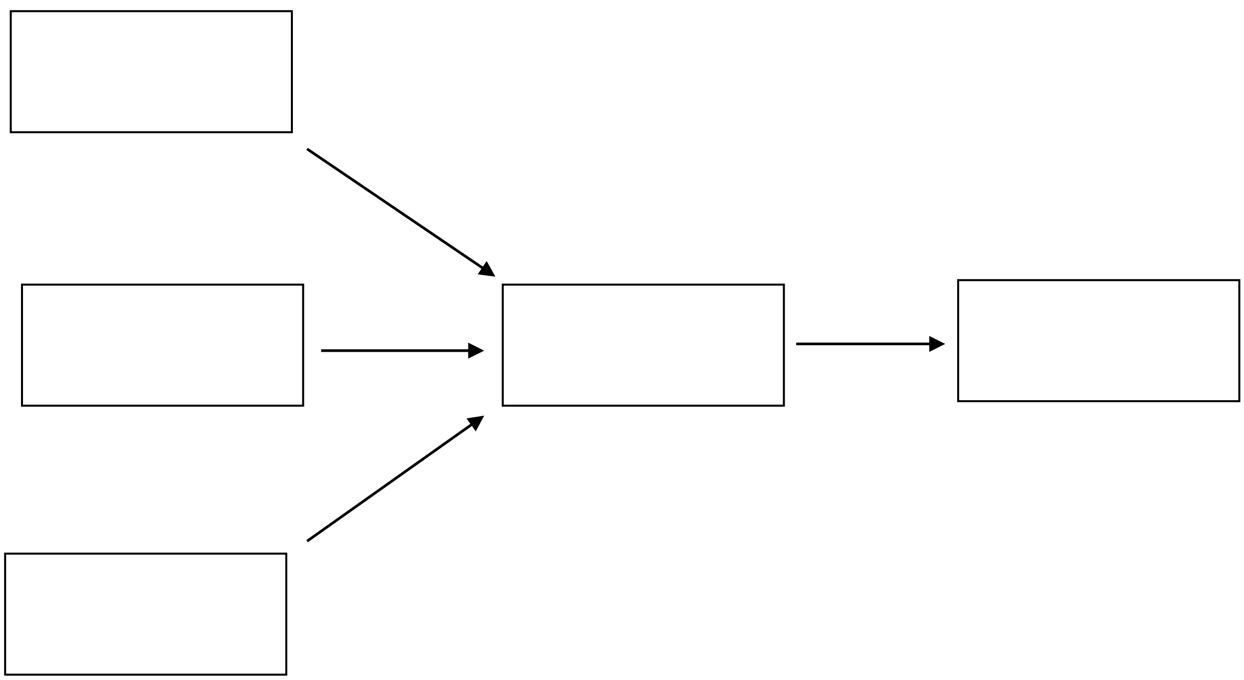
Complete the flow proof, including reasons for each step.

Given: $\overline{PR} \cong \overline{PQ}$

$\overline{PT} \cong \overline{PS}$

Show: $\angle PRT \cong \angle PQS$





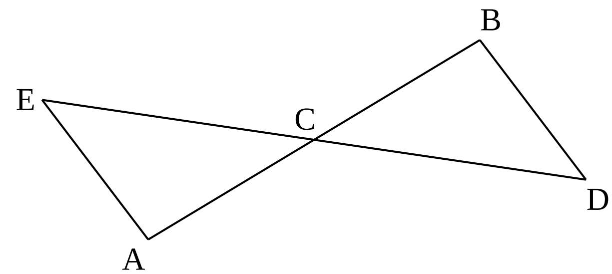
PROOF B

Write a proof of this statement.

Given: $\angle CDB \cong \angle CEA$

C is the midpoint of $\overline{ED}$

Show: $\overline{AE} \cong \overline{BD}$



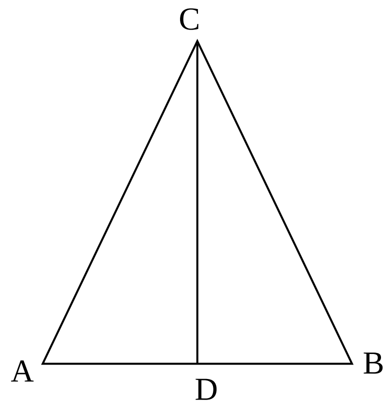
PROOF C

Write a proof of this statement.

Given: Δ ABC is isosceles with $\overline{AC} \cong \overline{BC}$

D is the midpoint of $\overline{AB}$

Show: $\overline{CD} \perp \overline{AB}$



Appendix D

Triangle Congruence Shortcut Conjectures

SSS - If three sides of one triangle are congruent to three sides of another triangle, then the triangles are congruent.

SAS - If two sides and the angle between them of one triangle are congruent to two sides and the angle between them of another triangle, then the triangles are congruent.

ASA - If two angles and the side between them of one triangle are congruent to two angles and the side between them of another triangle, then the triangles are congruent.

SAA - If two angles and a side not between them of one triangle are congruent to two angles and a side not between them of another triangle, then the triangles are congruent.

Appendix E

Student Survey on Using Congruent Triangles and Constructing Proofs

Please circle the level (0-10) that you feel best describes you.

Before we began this unit, how familiar were you with the triangle congruence theorems?

no knowledge										complete understanding
0	1	2	3	4	5	6	7	8	9	10

How would your rate your current knowledge of the congruence theorems now?

no knowledge										complete understanding
0	1	2	3	4	5	6	7	8	9	10

Before we began this unit, how familiar were you with constructing proofs?

no knowledge										complete understanding
0	1	2	3	4	5	6	7	8	9	10

How would your rate your ability to construct proofs now?

no knowledge										complete understanding
0	1	2	3	4	5	6	7	8	9	10

How would you rate your stress level while we were studying how to construct proofs?

no stress										very stressed
0	1	2	3	4	5	6	7	8	9	10

Appendix F

Sudoku Rules

Sudoku are easy to learn yet highly addictive language-independent logic puzzles which have recently taken the whole world by storm. Using pure logic and requiring no math to solve, these fascinating puzzles offer endless fun and intellectual entertainment to puzzle fans of all skills and ages.

The Classic Sudoku is a number placing puzzle based on a 9x9 grid with several given numbers. The object is to place the numbers 1 to 9 in the empty squares so that each row, each column and each 3x3 box contains the same number only once.

Sudoku puzzles come in endless number combinations and range from very easy to extremely difficult taking anything from five minutes to several hours to solve. Sudoku puzzles also come in many variants, each variant looking differently and each variant offering a unique twist of brain challenging logic.

However, make one mistake and you’ll find yourself stuck later on as you get closer to the solution… Try these puzzles, and see if you can solve them too!

Classic Sudoku

Each puzzle consists of a 9x9 grid containing given clues in various places. The object is to fill all empty squares so that the numbers 1 to 9 appear exactly once in each row, column and 3x3 box.

	9	3	6	2	8	1	4	
	6						5	
	3			1			9	
	5		8		2		7	
	4			7			6	
	8						3	
	1	7	5	9	3	4	2	

2	7	1	9	5	4	6	8	3
5	9	3	6	2	8	1	4	7
4	6	8	1	3	7	2	5	9
7	3	6	4	1	5	8	9	2
1	5	9	8	6	2	3	7	4
8	4	2	3	7	9	5	6	1
9	8	5	2	4	1	7	3	6
6	1	7	5	9	3	4	2	8
3	2	4	7	8	6	9	1	5

Mini Sudoku

Each puzzle consists of a 4x4 or 6x6 grid containing given clues in various places. The object is to fill all empty squares so that the numbers 1 to 4 (for 4x4 puzzles) or 1 to 6 (for 6x6 puzzles) appear exactly once in each row, column and box.

Appendix G

Sudoku Class Discussion

Sudoku – Pure Logic

Justifying Each Step

	A	*B*	*C*	*D*	*E*	*F*
a	3			1		5
b				3		
c		6			2	
d		5			3	
e			6			
f	5		2			6

2→Ba, 6→Ea, 4→Ca

2→De, 5→Dc, 6→Dd, 4→Df

2→Fb, 4→Eb

5→Ee, 1→Ef, 3→Fe

2→Ad, 6→Ab

5→Cb, 1→Bb

3→Bf, 4→Be, 1→Ae

4→Ac, 3→Cc, 1→Cd

4→Fd, 1→Fc

Appendix H

Sudoku Partner Practice

Sudoku – Pure Logic

Justify each step.

				3	
	3		2		5
		3		4	1
1	5		3		
3		5		6	
	6				

Appendix I

Sudoku Individual Practice

Sudoku – Pure Logic

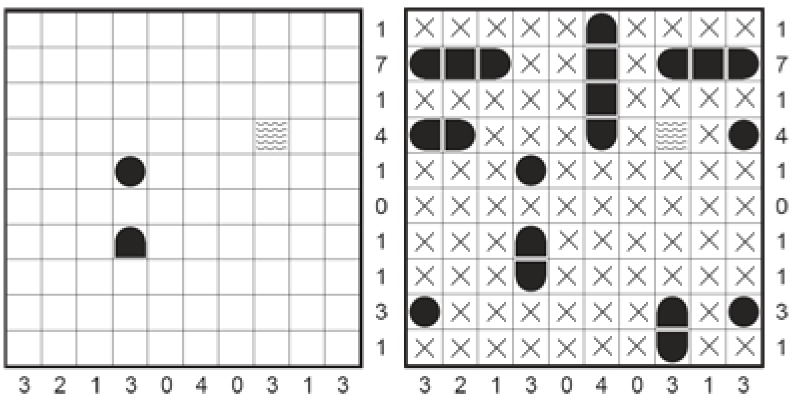
Justify each step.

4					
3					1
		4		2	
	6		5		
2					3
					2

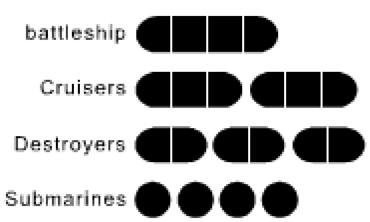
Appendix J

Battleships Instructions

Each Battleship puzzle represents an ocean with a hidden fleet of ships, which may be oriented horizontally or vertically within the grid such that no ship touches another, not even diagonally. The numbers on the right and on the bottom of the grid show how many squares in the corresponding row and column are occupied by ship segments. Occasionally some squares may contain given ship or water segments to help start the puzzle. The object is to discover where all ships are located.



Classic Battleship puzzles come in various sizes, with different fleets for each size. For example, the fleet of a Classic Battleship 10x10 puzzle consists of one battleship, two cruisers, three destroyers and four submarines.



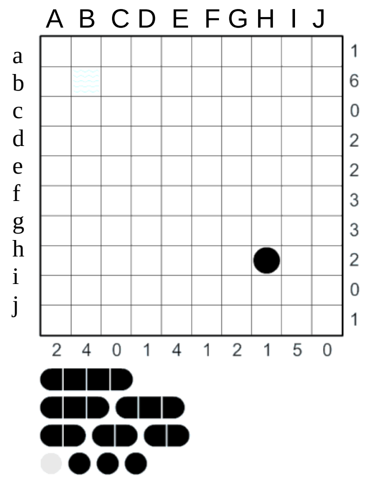
Submarines consist of a single round segment, destroyers have two end segments, cruisers have two end segments and a middle segment, and the battleship is constructed of two end segments and two middle segments. Any remaining squares in the grid contain water segments, which are shown as a symbol of water or as an “X”.

Appendix K

Battleships Class Discussion

Battleships – Pure Logic

Justifying Each Step



Mark water in row c, row I, column C, column J, the remainder of column H, and around the submarine.

Since row b must have 6 hits, there must be a cruiser in columns D-F or E-G or a battleship in D-G → shade Eb and Fb. Now mark water in the remainder of column F. Also mark water Da-Ga.

Row b must also include a hit at either Ab, Bb, or both → mark water in Aa and Ba.

Row a must have a hit at Ia. Then there must be water at Ib.

Now we have the battleship located at Db-Gb and a destroyer at Ab-Bb.

Mark water in the remainder of column D.

Since column I must have 5 hits, there must be a cruiser located at Id-If and a submarine at Ij. Mark water in the remainder of row j.

Since row g must have 3 hits, there must be a destroyer located at Ag-Bg and another hit at Eg.

Now mark water at Af-Bf and Ah-Bh and the remainder of column A.

There must now be hits at Eh and Ef, revealing a cruiser. Mark water in the remainder of column E.

There must be a submarine at Gf. Mark water in the remainder of column G.

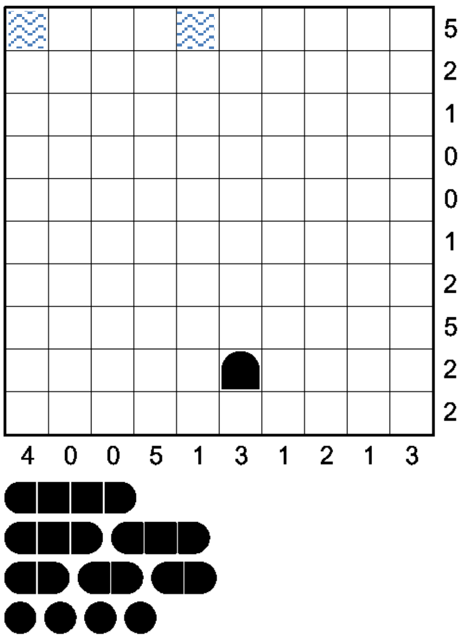
The final destroyer must now be located at Bd-Be.

Appendix L

Battleships Partner Practice

Battleships – Pure Logic

Justify each step.

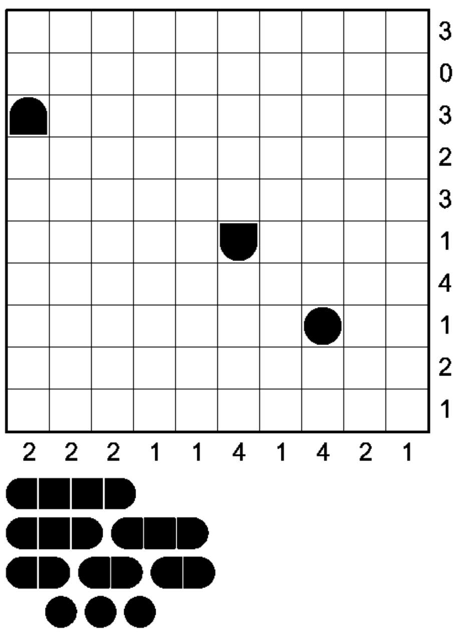


Appendix M

Battleships Individual Practice

Battleships – Pure Logic

Justify each step.



Appendix N

Hashi Instructions

Following the footsteps of Sudoku, Kakuro and other Number Logic puzzles, Hashi is one more family of easy to learn addictive logic puzzles which were invented in Japan. Using pure logic and requiring no math to solve, these fascinating puzzles offer endless fun and intellectual entertainment to puzzle fans of all skills and ages.

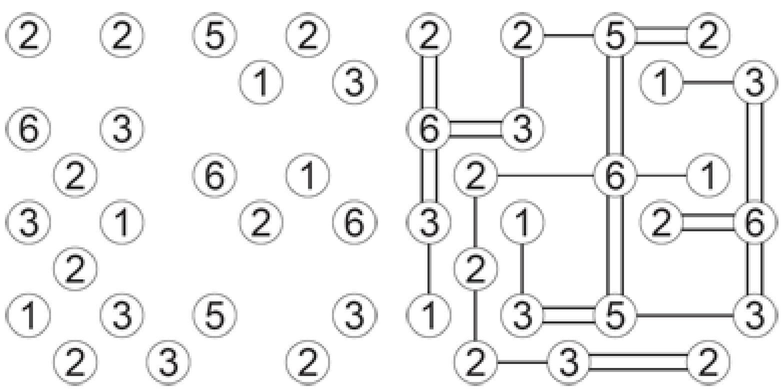
Hashi is a bridge-connecting puzzle. Unlike other logic puzzles, Hashi are solved by connecting islands with bridges according to the rules so that there is a continuous path between all islands.

Hashi puzzles come in many sizes and range from very easy to extremely difficult taking anything from five minutes to several hours to solve. However, make one mistake and you'll find yourself stuck later on as you get closer to the solution...

If you like Sudoku, Kakuro and other logic puzzles, you will love Conceptis Hashi as well!

Classic Hashi

Each puzzle is based on a rectangular arrangement of circles, where each circle represents an island and the number in each island tells how many bridges are connected to it. The object is to connect all islands according to the number of bridges so that there are no more than two bridges in the same direction and there is a continuous path connecting all islands together. Bridges can only be vertical or horizontal and are not allowed to cross islands or other bridges.

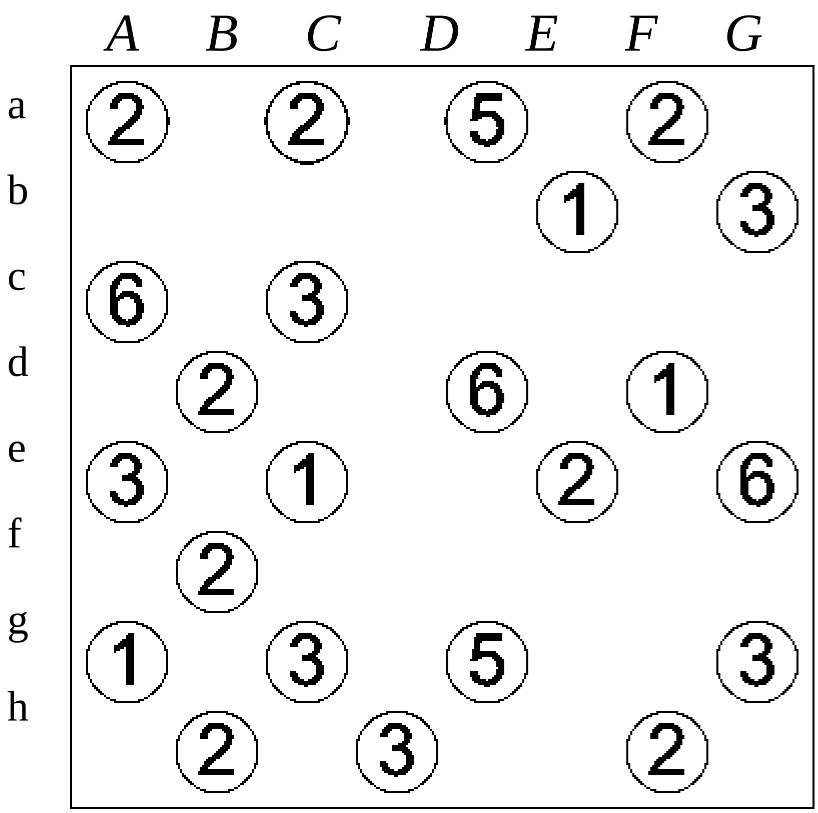


Appendix O

Hashi Class Discussion

Hashi – Pure Logic

Justifying Each Step



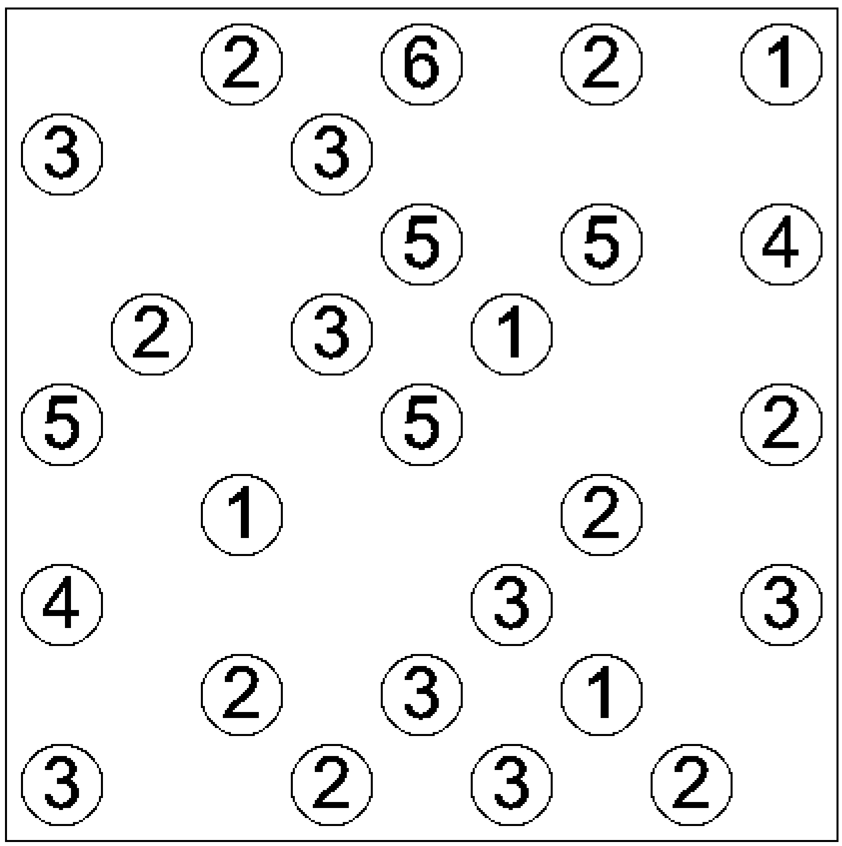
6 bridges at Ac
6 bridges at He
3rd bridge at Hg
2nd-5th bridge at Eg
3rd bridge at Cg
3rd bridge at Ae
2 bridges at Bf
2nd bridge at Bh
2nd-3rd bridge at Dh
2nd bridge at Bd
3rd-6th bridges at Ed
1 bridge at Fb
2 bridges at Ga
2 bridges at Ca

Appendix P

Hashi Partner Practice

Hashi – Pure Logic

Justify each step.

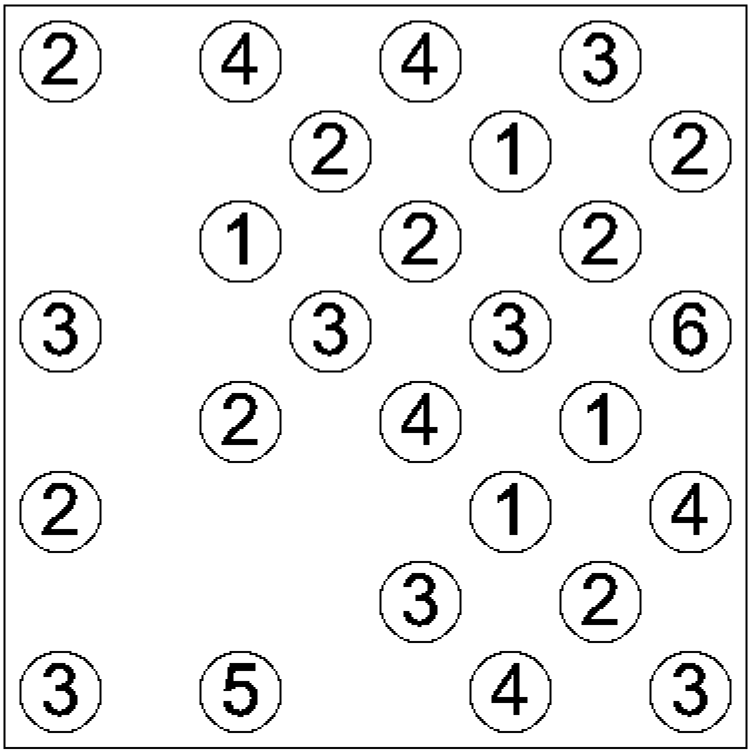


Appendix Q

Hashi Individual Practice

Hashi – Pure Logic

Justify each step.



Appendix R

Pic-a-Pix Instructions

Pic-a-Pix are exciting Picture Logic puzzles that form whimsical pixel-composed pictures when solved. Challenging, deductive and artistic, this original Japanese invention offers the ultimate mix of logic, art and fun while providing solvers with many hours of mentally stimulating entertainment.

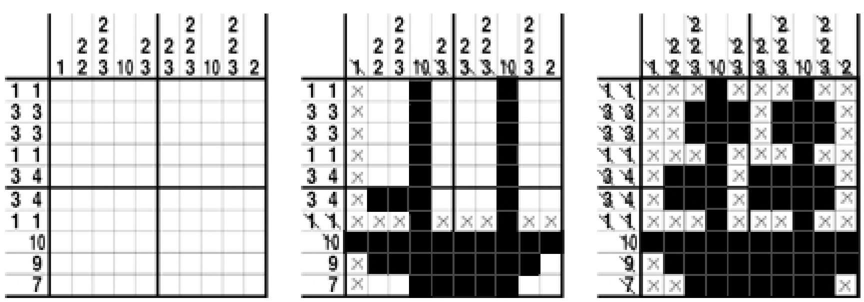
Pic-a-Pix is a block-placing puzzle based on a grid with a pixilated picture hidden inside. Using logic alone, the solver determines which squares are painted and which should remain empty until the hidden picture is completely exposed.

Pic-a-Pix puzzles come in B&W and color, and are available in many sizes and difficulty levels taking anything from five minutes to several hours to solve. However, make one mistake and you’ll find yourself stuck later on as you get closer to the solution...

If you like Sudoku, Kakuro and other logic puzzles, you will love Conceptis Pic-a-Pix as well!

B&W Pic-a-Pix

Each puzzle consists of a blank grid with clues on the left of every row and on the top of every column. The object is to reveal a hidden picture by painting blocks in each row and column so their length and sequence corresponds to the clues, and there is at least one empty square between adjacent blocks.

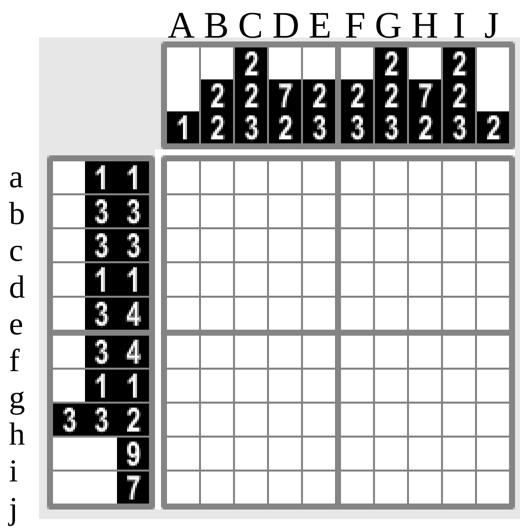


Appendix S

Pic-A-Pix Class Discussion

Pic-a-Pix – Pure Logic

Justifying Each Step



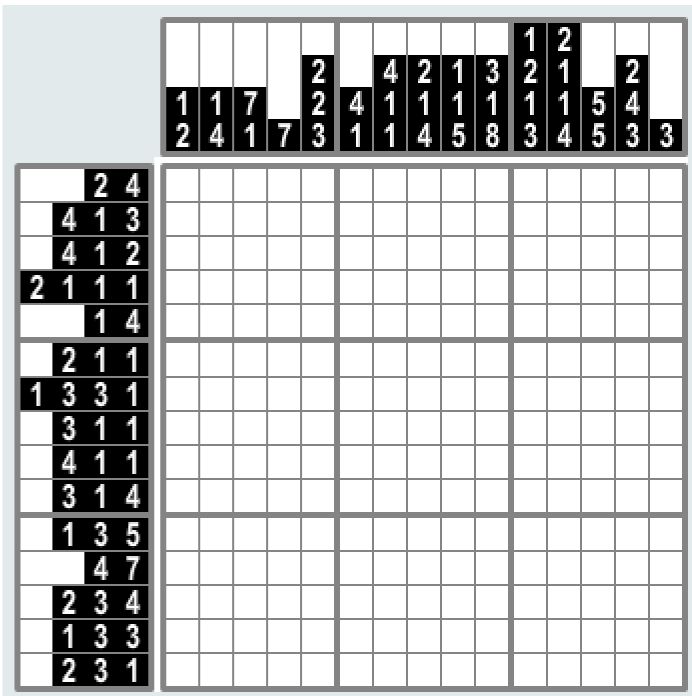
Complete row h and columns D & H
Fill in Ej-Gj
Fill in Bi-Ii
Fill in Cj, Ij and Ji
Mark empty squares in rows a,d, g, and column j
Fill in Gb-Gc and Ge-Gf
Fill in Ib-Ic and Ie-If
Fill in Fe-Ff
Fill in Be-Ce and Bf-Cf
Fill in Eb-Ec
Fill in Cb-Cc

Appendix T

Pic-A-Pix Partner Practice

Pic-a-Pix – Pure Logic

Justify each step.

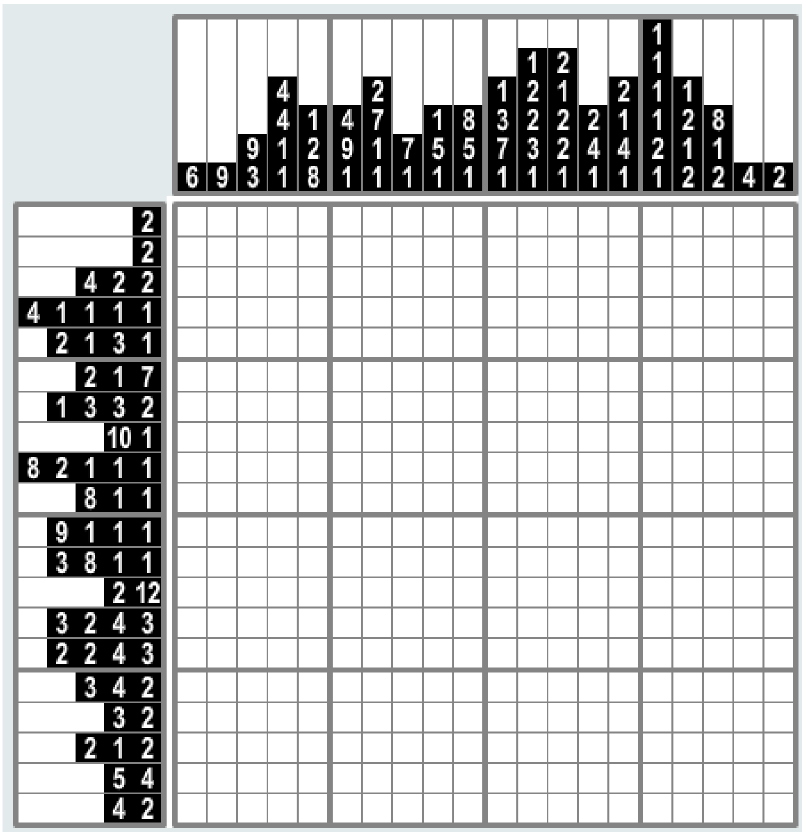


Appendix U

Pic-A-Pix Individual Practice

Pic-a-Pix – Pure Logic

Justify each step.



Appendix V

Hitori Instructions

Following the footsteps of Sudoku and Kakuro, Hitori are yet another type of easy to learn addictive logic puzzle which was invented in Japan. Using pure logic and requiring no math to solve, these fascinating puzzles offer endless fun and intellectual entertainment to puzzle fans of all skills and ages.

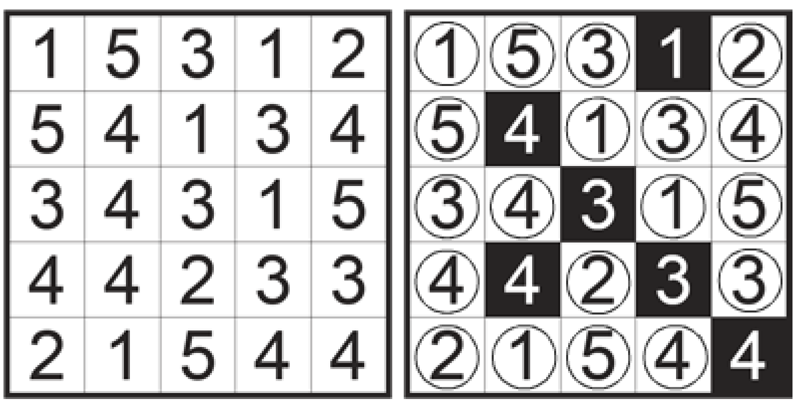
Hitori is a number-elimination puzzle. Unlike Sudoku and Kakuro, Hitori puzzles start with all the numbers in the grid and your task is to eliminate some of them according to the rules.

Hitori puzzles come in many sizes and range from very easy to extremely difficult taking anything from five minutes to several hours to solve. However, make one mistake and you'll find yourself stuck later on as you get closer to the solution...

If you like Sudoku, Kakuro and other logic puzzles, you will love Conceptis Hitori as well!

Classic Hitori

Each puzzle consists of a square grid with numbers appearing in all squares. The object is to shade squares so that the numbers don't appear in a row or column more than once. In addition, shaded squares must not touch each other vertically or horizontally while all un-shaded squares must create a single continuous area.



Appendix W

Hitori Class Discussion

Hitori – Pure Logic

Justifying Each Step

	A	*B*	*C*	*D*
a	2	2	2	4
b	1	4	2	3
c	2	3	2	1
d	3	4	1	2

Shade Aa and Ca
Shade Cc
Shade Bd

Appendix X

Hitori Partner Practice

Hitori – Pure Logic

Justify each step.

3	6	2	6	5	6
5	1	3	6	2	2
5	6	1	6	4	3
6	2	6	5	1	5
1	1	4	1	3	5
3	3	5	2	6	5

Appendix Y

Hitori Individual Practice

Hitori – Pure Logic

Justify each step.

6	5	7	4	4	8	5	6
5	4	8	6	1	3	8	7
8	1	2	7	7	3	5	5
1	5	8	7	6	7	2	4
6	7	1	2	5	1	1	3
2	5	5	5	8	7	4	7
4	6	3	5	4	2	1	8
4	8	4	5	2	6	5	1

Appendix Z

Nurikabe Instructions

Following the footsteps of Sudoku, Kakuro and other Number Logic puzzles, Nurikabe is one more family of easy to learn addictive logic puzzles which were invented in Japan. Using pure logic and requiring no math to solve, these fascinating puzzles offer endless fun and intellectual entertainment to puzzle fans of all skills and ages.

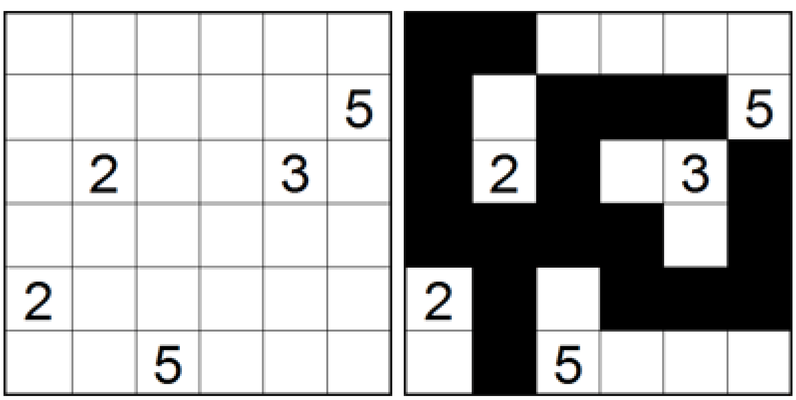
Nurikabe is an island-forming puzzle. Unlike other logic puzzles, Nurikabe are solved by partitioning between clues with walls according to the rules so that all islands are isolated and there is a continuous path to all walls.

Nurikabe puzzles come in many sizes and range from very easy to extremely difficult taking anything from five minutes to several hours to solve. However, make one mistake and you'll find yourself stuck later on as you get closer to the solution...

If you like Sudoku, Kakuro and other logic puzzles, you will love Conceptis Nurikabe as well!

Classic Nurikabe

Each puzzle consists of a grid containing clues in various places. The object is to create islands by partitioning between clues with walls so that the number of squares in each island is equal to the value of the clue, all walls form a continuous path and there are no wall areas of 2x2 or larger. Each island must contain one clue and be isolated from other islands horizontally and vertically.



Appendix AA

Nurikabe Class Discussion

Nurikabe– Pure Logic

Justifying Each Step

	A	*B*	*C*	*D*	*E*	*F*
a						
b		1				
c	5		3			
d						
e			2			6
f						

Fill in Aa-Ca, Ab, Cb, and Bc
Fill in Da
Fill in Bd-Be
Fill in Cf-Df
Fill in Cd-Dd
Fill in Ea-Ef

Appendix BB

Nurikabe Partner Practice

Nurikabe – Pure Logic

Justify each step.

3						2	
		2					
						6	
		2					
					2		3
	3					1	
			3				

Appendix CC

Nurikabe Individual Practice

Nurikabe – Pure Logic

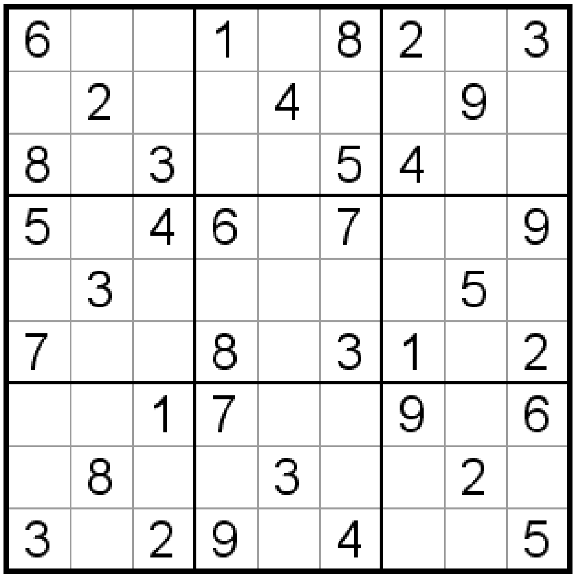
Justify each step.

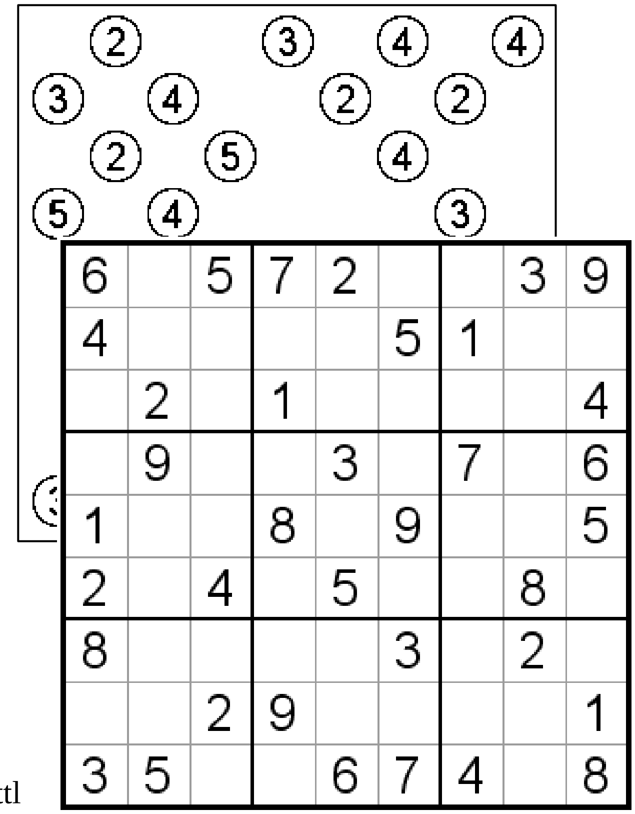
						2			
4		2					2		
									4
						3			
3			2				1		
						3			
		1							
						3			
			4				4		
									6

Appendix DD

Logic Puzzle Practice Problems

Sudoku





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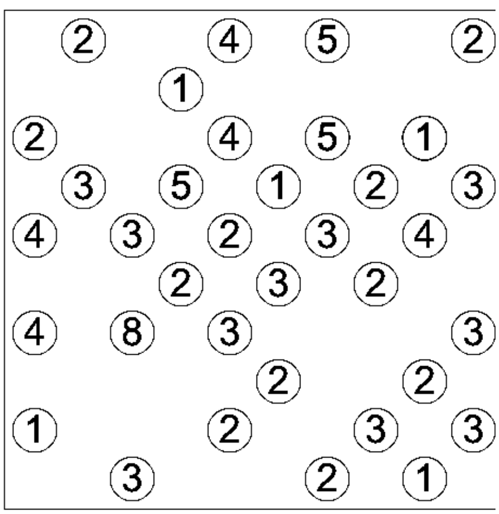
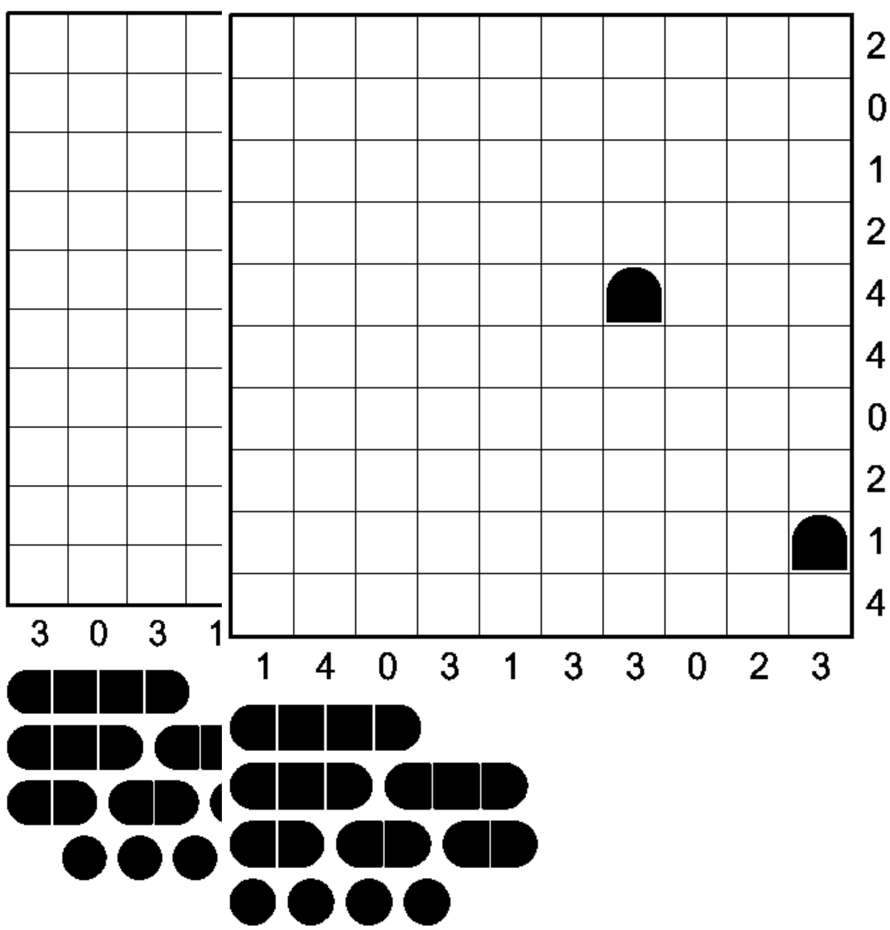


Hitori



5	2	8	8	3	4	7	1
2	1	6	4	6	5	5	7
8	4	5	4	7	3	4	1
3	4	4	6	6	2	3	8
6	2	7	2	1	4	8	3
1	6	6	3	8	7	5	2
5	3	1	4	4	2	6	3
4	5	8	7	8	1	3	4

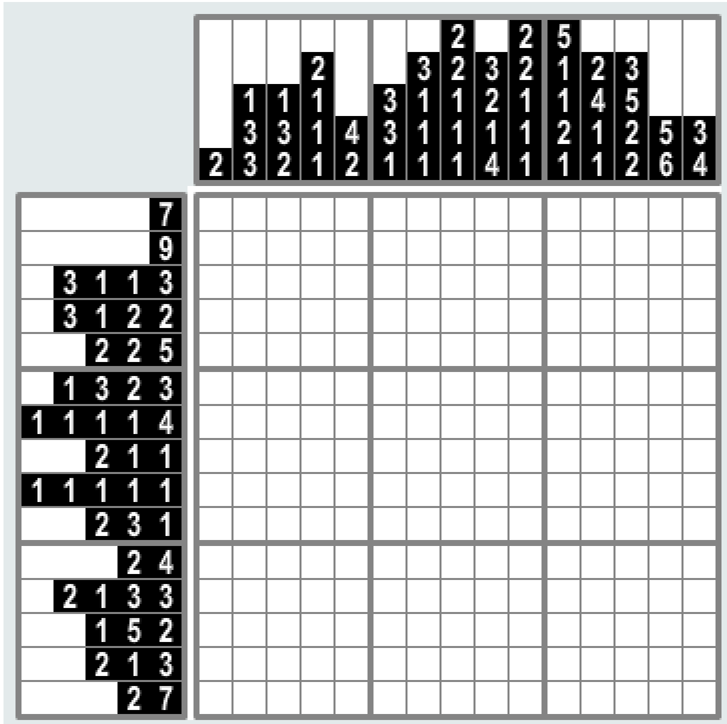
Nurikabe

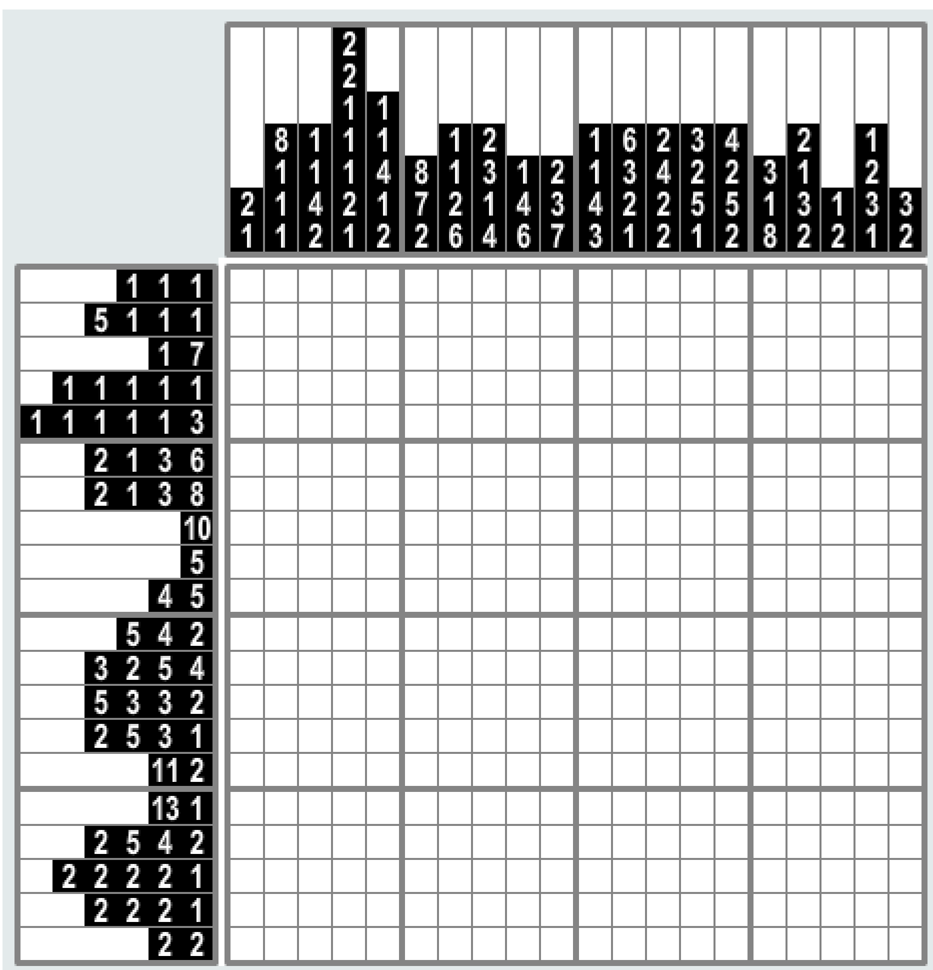
5	3	5	7	10	10	6	4	2	8
9	1	10	9	8	2	7	4	3	5
4	6	7	1	2	1	10	4	5	3
6	1	4	1	7	8	7	5	1	2
4	9	4	10	4	7	3	7	8	6
7	5	6	9	5	4	2	8	3	2
3	10	4	8	6	7	5	3	7	1
2	8	9	8	5	3	8	2	6	7
10	5	2	4	8	1	8	6	9	7
2	4	7	6	3	2	9	2	10	7

	2			2				8	
					1		7		
2									
				3					
					5				
		1							
			8						
							2		
					7				

3					7				
		1							
					2				2
	4			2					
4									
					3				
			3						
							1		
					3				
			5						8

Pic-A-Pix


2
1 3 3
1 3 2
2 1 1 1
4 2
3 3 1
3 1 1 1
2 2 1 1 1
3 2 1 4
2 2 1 1 1
5 1 1 2 1
2 4 1 1
3 5 2 2
5 6
3 4
7
9
3 1 1 3
3 1 2 2
2 2 5
1 3 2 3
1 1 1 1 4
2 1 1
1 1 1 1 1
2 3 1
2 4
2 1 3 3
1 5 2
2 1 3
2 7


2 1
8 1 1 1
1 1 4 2
2 2 1 1 1 2 1
1 1 4 1 2
8 7 2
1 1 2 6
2 3 1 4
1 4 6
2 3 7
1 1 4 3
6 3 2 1
2 4 2 2
3 2 5 1
4 2 5 2
3 1 8
2 1 3 2
1 2
1 2 3 1
3 2
1 1 1
5 1 1 1
1 7
1 1 1 1 1
1 1 1 1 1 3
2 1 3 6
2 1 3 8
10
5
4 5
5 4 2
3 2 5 4
5 3 3 2
2 5 3 1
11 2
13 1
2 5 4 2
2 2 2 2 1
2 2 2 1
2 2