Fundamental Solutions for Multivariate Difference Equations and Applications to Cardinal Interpolation

by

Carl de Boor
Klaus Höllig
Sherman Riemenschneider

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Fundamental solutions for multivariate difference equations and applications to cardinal interpolation

Carl de Boor[1,2], Klaus Höllig[2,3] and Sherman Riemenschneider[4]

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ABSTRACT

Let $b : \mathbb{Z}^d \mapsto \mathbb{R}$ be a mesh-function with compact support. It is shown that the difference equation

$$\sum_{j \in \mathbb{Z}^d} b(k-j)a(j) = f(k), \quad k \in \mathbb{Z}^d,$$

has a bounded solution a if $|f(j)| = O((1+|j|)^{-n})$ for some exponent n which depends on b. This result is the discrete analogue of the existence of tempered fundamental solutions for partial differential operators with constant coefficients. It is applied to prove optimal convergence rates for interpolation with box-splines.

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1. Introduction

Let b be a function with compact support defined on the integer lattice $\mathbb{Z}^d$. We consider the difference equation

$$(\Delta) \qquad (b * a)(k) := \sum_{j \in \mathbb{Z}^d} b(k-j)a(j) = f(k), \quad k \in \mathbb{Z}^d.$$

Typical examples arise from discretization of partial differential equations and in interpolation problems.

Standard theory for (Δ) requires that the characteristic polynomial

$$(1) \qquad B(x) := \sum_j \exp(ijx)b(j)$$

not vanish. In this case, the difference equation is easily solved. Multiplying both sides of (Δ) by $\exp(ikx)$ and summing over k yields

$$BA = F$$

where, as in (1), we denote Fourier series by capital letters. It follows that $a(j)$ are the Fourier coefficients of F/B. In particular,

$$(2) \qquad \check{b}(j) := \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \frac{\exp(-ijx)}{B(x)}\, dx$$

is a fundamental solution of (Δ), i.e.

$$(3) \qquad \sum_j b(k-j)\check{b}(j) = \delta(k) := \begin{cases} 1, & \text{if } k = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$(4) \qquad a = \check{b} * f.$$

This proves the following well known

Theorem. *If $B(x) \neq 0$ for all x, then $\check{b}$, defined by (2), satisfies*

$$|\check{b}(j)| = O(\exp(-c|j|))$$

for some positive c, and, for bounded f, the unique bounded solution of (Δ) is given by (4).

If $B(x_0) = 0$, then $a(j) = \exp(ijx_0)$ is a nontrivial solution of the homogeneous system. Therefore, the standard theory which is based on (2) – (4) does not apply. However, also in this case, the system (Δ) still has a bounded solution provided the data decay sufficiently fast.

Theorem 1. *There exists a fundamental solution $\check{b}$ of (Δ) with*

$$|\check{b}(j)| = O((1+|j|)^m) \tag{5}$$

for some $m > 0$ which depends on b. Therefore, in view of (4), there exists a bounded solution a of (Δ) if $|f(j)| = O((1+|j|)^{-n})$ for some $n > m+d$.

As will become clear from the proof given in Section 2, this theorem is the discrete analogue of the well known result about the existence of tempered fundamental solutions for differential equations with constant coefficients. Some applications of Theorem 1 are discussed in Section 3.

2. Fundamental solutions

The difference equation (Δ) is the discrete analogue of the differential equation

$$p(D)\varphi = \psi \tag{D}$$

where p is a polynomial in the partial derivatives D_ν, $\nu = 1, \ldots, d$. Łojasiewicz [L59] showed that (D) has a tempered fundamental solution, i.e. there exists a tempered distribution $\check{p}$ such that

$$p(D)(\check{p} * \psi) = \psi$$

for any rapidly decreasing test-function ψ. He considered the more general case when p is an analytic function, and this settled the "division problem" posed by Schwartz. In the polynomial case, a simplified proof of Łojasiewicz' result was given by Hörmander [H58]. The main step in either proof is to show that the map

$$\varphi \mapsto p\varphi$$

is bounded below in the topology of the space of rapidly decreasing test-functions. The corresponding estimate applies in particular to periodic functions and, for this case, can be formulated as follows, using the standard norm

$$|G|_m := \sup_j (1+|j|)^m |g(j)|$$

for the periodic function $G(x) = \sum_j \exp(ijx) g(j)$.

Proposition. *For any r there exists an integer r' (necessarily $\geq r$) and a constant c so that*

$$|A|_r \leq c|BA|_{r'} \quad \text{for all } A \in C^\infty_{\text{periodic}}. \tag{6}$$

This proposition is easily deduced from the corresponding result in [H58]. Inequality (4.3) in conjunction with the remarks after Theorem 1 in that paper implies that, for any analytic function p,

$$|\varphi|_{r,s} \leq c|p\varphi|_{r',s'} \tag{7}$$

where

$$|\varphi|_{r,s} := \max_{\substack{|\alpha|\leq r \\ \ell\leq s}} \sup_{x\in\mathbb{R}^d} |x^\ell D^\alpha \varphi(x)|$$

and r', s' depend on r, s and p. To derive (6) from (7) let χ be a smooth cut-off function which is equal to 1 on $[-\pi,\pi]^d$ and note that

$$|A|_r \leq c'|\chi A|_{r,0} \leq cc'|\chi BA|_{r',0'} \leq c''|BA|_{r''}.$$

We now apply the Proposition to **prove Theorem 1**. By (6), the functional

$$\Lambda_0 : BA \mapsto a(0) \tag{8}$$

is well defined on $V := \{BA : A \in C^\infty_{\text{periodic}}\}$ and

$$|\Lambda_0 G| \leq c|G|_m, \quad G \in V. \tag{9}$$

By the Hahn-Banach Theorem, we can therefore think of Λ_0 as the restriction to V of a bounded linear functional Λ on the space of all functions G with $|G|_m < \infty$. Therefore, Λ can be represented in the form

$$\Lambda G = \sum_j \lambda(-j)g(j)$$

with $|\lambda(j)| = O((1+|j|)^m)$. In particular, for any compactly supported a,

$$a(0) = \sum_j \lambda(-j)\sum_k b(j-k)a(k),$$

hence, with $a = \delta(\cdot - \ell)$,

$$\delta(-\ell) = \sum_j \lambda(-j)b(j-\ell) = \sum_j b(-\ell-j)\lambda(j),$$

or, $\delta = b * \lambda$. This completes the proof of Theorem 1.

In general, little can be said about the dependence of m on b. However, in the univariate case ($d = 1$), a more precise statement is possible.

Theorem 2. *If $d = 1$, then Theorem 1 holds with $m + 1$ equal to the highest multiplicity of zeros of B. In particular, if B has only simple zeros, there exists a bounded fundamental solution, and consequently (Δ) has a bounded solution for data f which satisfy $\sum_j |f(j)| < \infty$.*

Proof. Assume that $x_0, \ldots, x_n$ are the zeros of B, repeated according to their multiplicity. Denote by $H(x_0, \ldots, x_n; f)$ the Hermite interpolant of f at the points x_ν by a trigonometric polynomial of the form $\sum_{\nu=0}^{n} \exp(i\nu\cdot)c(\nu)$. Then, a suitable functional Λ can be explicitly defined by

$$\Lambda G := \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{G(x) - H(x_0, \ldots, x_n; G)(x)}{B(x)}\, dx. \tag{10}$$

Indeed, since $H(x_0, \ldots, x_n; BA) = 0$ for any smooth function A, $\Lambda(BA) = a(0)$. To estimate the growth of

$$\lambda(j) = \Lambda \exp(-ij\cdot)$$

as $|j| \to \infty$, we substitute $G(x) = \exp(-ijx)$ in (10), set $z := \exp(ix)$ and write $G - H$ explicitly as the remainder term in polynomial interpolation. Writing the resulting expression as a contour integral yields

$$\lambda(j) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{[z_0, \ldots, z_n, z](\,)^{-j}(z - z_0)\cdots(z - z_n)}{\tilde{B}(z)} \frac{dz}{z}$$

where $z_\nu := \exp(ix_\nu)$, $\tilde{B}(z) := B(x)$ and $[\ldots]$ denotes the divided difference. This contour integral is computed by summing the residues inside (outside) the unit circle for $j < 0$ ($j \geq 0$). We consider only the case $j < 0$, the other being similar. Assume that $z\tilde{B}(z) = (z - z_*)^{r+1}R(z)$ with $R(z_*) \neq 0$ and $|z_*| < 1$. Then,

$$\text{residue}_{|z_*} = \frac{1}{r!} \frac{d^r}{dz^r} \left(\frac{[z_0, \ldots, z_n, z](\,)^{-j}(z - z_0)\cdots(z - z_n)}{R(z)} \right)_{|z=z_*}.$$

It is well known that the divided difference

$$[M]f$$

of f at the multipoint set M can be written explicitly as a sum

$$[M]f = \sum_{z \in M} \sum_{s < \#z} c_{z,s} D^s f(z)$$

with $\#z$ the multiplicity of z in M and $c_{z,s}$ a rational function in the elements of M which is independent of f. In particular,

$$[z_0, \ldots, z_n, z]()^{-j} = c(z)z^{-j} + \sum_{\nu}\sum_{\mu=0}^{m_\nu}(-j)(-j-1)\cdots(-j-\mu+1)c_{\nu,\mu}(z)z_\nu^{-j-\mu}$$

where $m_\nu + 1$ is the multiplicity of z_ν and $c, c_{\nu,\mu}$ are functions which do not depend on j. It follows that

$$\text{residue}_{|z_*} = O((1+|j|)^{\max_\nu m_\nu})$$

as $|j| \to \infty$.

To illustrate the main result of this section, we explicitly construct a fundamental solution for the discrete Laplace operator in two dimensions, i.e. we consider the difference equation

$$-4a(\nu,\mu) + a(\nu-1,\mu) + a(\nu+1,\mu) + a(\nu,\mu-1) + a(\nu,\mu+1) = f(\nu,\mu), \quad (\nu,\mu) \in \mathbb{Z}^2. \tag{11}$$

In this case

$$\begin{aligned} B(x) &= -4 + (e^{iu} + e^{-iu}) + (e^{iv} + e^{-iv}) \\ &= -4(\sin^2\frac{u}{2} + \sin^2\frac{v}{2}), \qquad x = (u,v). \end{aligned}$$

Since B has only an isolated zero at the origin, the functional Λ can be defined by

$$\Lambda G := -\frac{1}{4(2\pi)^2}\int_{[-\pi,\pi]^2} \frac{G(u,v) - G(0,0)}{\sin^2\frac{u}{2} + \sin^2\frac{v}{2}}\,dudv.$$

That this integral exists and is finite follows from the estimate

$$\begin{aligned} |\Lambda G| &\le \frac{1}{16}\int_{[-\pi,\pi]^2} \frac{|G(u,v) - G(0,0)|}{u^2+v^2}\,dudv \\ &\le \frac{1}{16}\int_{[-\pi,\pi]^2} \frac{\min\{2\|G\|_\infty, |u|\|\partial_u G\|_\infty + |v|\|\partial_v G\|_\infty\}}{u^2+v^2}\,dudv \\ &\le \frac{1}{16}\int_0^{2\pi}\int_0^{\sqrt{2}\pi} \min\{2\|G\|_\infty, \rho(\|\partial_u G\|_\infty + \|\partial_v G\|_\infty)\}\,\frac{d\rho}{\rho}d\theta. \end{aligned}$$

Substituting $G(x) = e^{ijx}$, $j = (\nu,\mu)$, we find that

$$\lambda(j) = \frac{-1}{4(2\pi)^2}\int_{[-\pi,\pi]^2} \frac{e^{i\nu u}e^{i\mu v} - 1}{\sin^2\frac{u}{2} + \sin^2\frac{v}{2}}\,dudv$$

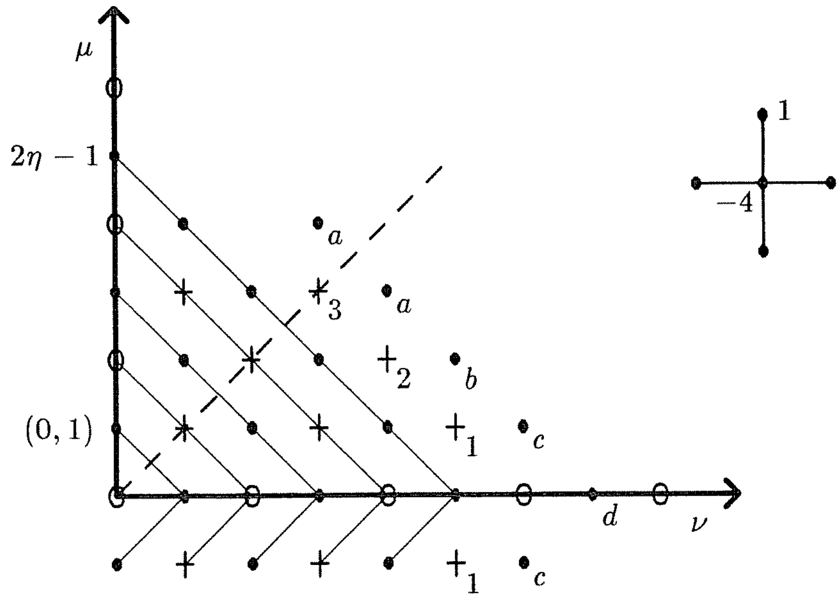

Figure 1. Construction of a Green's function for the discrete Laplacian

with the estimate

$$\begin{aligned} |\lambda(j)| &\le \frac{1}{16}\int_0^{2\pi}\int_0^{\sqrt{2}\pi} \min\{2, \rho(|\nu| + |\mu|)\}\,\frac{d\rho}{\rho}\,d\theta \\ &\le \frac{1}{16}\int_0^{2\pi}\int_0^{(|\nu|+|\mu|)\sqrt{2}\pi} \min\{2, \rho\}\,\frac{d\rho}{\rho}\,d\theta \\ &= O(1 + \log|j|). \end{aligned}$$

The exact value of $\lambda(j)$ is a little more difficult to obtain. We rewrite the integral as

$$\lambda(\nu, \mu) = \frac{-1}{2(2\pi)^2}\int_{[-\pi,\pi]^2} \frac{\cos\nu u \cos\mu v - 1}{2 - \cos u - \cos v}\,dv\,du. \tag{12}$$

Clearly, $\lambda(0) = 0$ and $\lambda(\nu, \mu)$ is symmetric about the lines $\nu = 0$, $\mu = 0$, and $\nu = \mu$. Since $a = \lambda$ is the solution of the difference equation (11) for $f = \delta$,

$$-4\lambda(\nu,\mu) + \lambda(\nu-1,\mu) + \lambda(\nu+1,\mu) + \lambda(\nu,\mu-1) + \lambda(\nu,\mu+1) = \delta(\mu,\nu). \tag{13}$$

Therefore, all values of $\lambda(\nu, \mu)$ can be computed from the knowledge of $\lambda(2\eta, 0)$, $\eta = 0, 1, \ldots,$ and (13) (cf. Figure 1). Indeed, (13) for $(\nu, \mu) = (0, 0)$ and symmetry yields $\lambda(1, 0) = 1/4$. Now suppose that $\lambda(\nu, \mu)$ is known for $|\nu| + |\mu| \le 2\eta - 1$ (shaded region). Given the value of $\lambda(2\eta, 0)$, we can compute $\lambda(2\eta - 1, 1)$ using (13) and symmetry (points labeled 1). Then, (13) is used to compute successively $\lambda(2\eta - s, s)$, $s = 2, \ldots, \eta$ (points labeled 2, 3). Finally, we compute $\lambda(\eta + 1 + s, \eta - s)$, $s = 0, \ldots, \eta$, using symmetry for $s = 0$ and $s = \eta$ (points labeled a, b, c, d).

Therefore, we need to compute only the integral

$$\begin{aligned}
\lambda(2\eta, 0) &= \frac{-1}{2\pi^2} \int_0^\pi \int_0^\pi \frac{\cos 2\eta u - 1}{2 - \cos u - \cos v}\, dv\, du \\
&= \frac{-1}{2\pi} \int_0^\pi \frac{\cos 2\eta u - 1}{\sqrt{(2 + \cos u)^2 - 1}}\, du \\
&= \frac{-1}{2\pi} \int_{-1}^1 \frac{T_{2\eta}(x) - 1}{\sqrt{(2 + x)^2 - 1}} \frac{dx}{\sqrt{1 - x^2}} \\
&= \frac{-1}{2\pi} \int_{-1}^1 \frac{T_{2\eta}(x) - 1}{x + 1} \frac{dx}{\sqrt{4 - (x + 1)^2}}
\end{aligned}$$

where $T_{2\eta}$ is the Chebyshev polynomial. We may determine the coefficients in the expansion

$$T_{2\eta}(x) - 1 = \sum_{l=1}^{2\eta} t_l (x + 1)^l$$

as

$$t_l = \frac{(-1)^l 2^{l-1}}{(2l - 1)!} \prod_{s=0}^{l-1} \left((2\eta)^2 - s^2\right), \qquad l = 1, \ldots, 2\eta,$$

from the fact that $T_{2\eta}$ satisfies the differential equation

$$(1 - x^2) y'' - x y' + (2\eta)^2 y = 0.$$

With the substitution $x + 1 = 2 \sin\theta$, we have that

$$\begin{aligned}
\lambda(2\eta, 0) &= \frac{-1}{2\pi} \sum_{l=1}^{2\eta} t_l 2^{l-1} \int_0^{\frac{\pi}{2}} (\sin\theta)^{l-1}\, d\theta \\
&= \frac{-\sqrt{\pi}}{8} \sum_{l=1}^{2\eta} t_l 2^l \frac{\Gamma\left(\frac{l}{2}\right)}{\Gamma\left(\frac{l+1}{2}\right)}.
\end{aligned}$$

3. Applications to Cardinal Interpolation

In this Section we apply Theorem 1 to interpolation with translates of box-splines which generalizes recent work by Chui, Diamond and Raphael [CDR87].

Box-spline interpolation is a special case of cardinal interpolation. In **cardinal interpolation**, one is given a compactly supported function b on $\mathbb{R}^d$ (and not just on $\mathbb{Z}^d$), and is to construct a sequence a for which the function

$$b * a \;=\; \sum_{j\in\mathbb{Z}^d} b(\cdot - j)a(j)$$

agrees on $\mathbb{Z}^d$ with a given function f. Hence the problem is equivalent to solving the difference equation

$$b_| * a = f_|,$$

with

$$f_| := f_{|\mathbb{Z}^d}.$$

If $\check{b}$ is a fundamental solution of this difference equation, and L is defined by

$$L := b * \check{b} = \sum_{j\in\mathbb{Z}^d} b(\cdot - j)\check{b}(j),$$

then

$$\mathcal{L}f := L * f_| = \sum_{j\in\mathbb{Z}^d} L(\cdot - j)f(j)$$

is a cardinal interpolant to f, provided it makes sense. Specifically, if

$$\check{b}(j) = O((1 + |j|)^m), \tag{14}$$

then $\mathcal{L}f$ is defined and bounded for any f with

$$f(x) = O((1 + |x|)^{-n}) \quad \text{for some } n > m + d. \tag{15}$$

We will call such L a **fundamental solution of order** m of the cardinal interpolation problem.

We have particular interest in the case when b is the box-spline M_Ξ. The box-spline $M_\Xi : \mathbb{R}^d \mapsto \mathbb{R}$ can be defined by its Fourier transform

$$\hat{M}_\Xi(x) := \prod_{\xi\in\Xi} \operatorname{sinc}(\xi x/2)$$

where Ξ is a multiset of vectors in $\mathbb{Z}^d$ and $\operatorname{sinc}(t) := \sin(t)/t$. As is apparent from this definition, the box-spline is a natural generalization of the univariate cardinal B-spline M_r which corresponds to

$$\Xi = \{\underbrace{1, \ldots, 1}_{r \text{ times}}\}.$$

The basic properties of M_Ξ are discussed in [BH83]. For the purpose of this paper we merely note that M_Ξ is a positive piecewise polynomial of degree $\#\Xi - d$ with compact support which is continuous if every subset of $\#\Xi - 1$ vectors from Ξ spans $\mathbb{R}^d$.

In the following we assume that M_Ξ is continuous. Theorem 1 provides the following

Corollary 1. *Define the fundamental spline*

$$L_\Xi := \sum_{j \in \mathbb{Z}^d} M_\Xi(\cdot - j)\check{b}(j),$$

with $\check{b}$ a fundamental solution of (Δ) for $b = M_\Xi$ and satisfying (5). Then, for any continuous function f with $|f(j)| = O((1+|j|)^{-n})$ (in particular any function with compact support) there exists a bounded cardinal box-spline interpolant

$$\mathcal{L}_\Xi f = \sum_{j \in \mathbb{Z}^d} L_\Xi(\cdot - j) f(j).$$

Schoenberg, in his classical results on univariate cardinal interpolation (reported, e.g., in [S73]), assumes that the characteristic polynomial $B_\Xi := \sum_j \exp(ij\cdot) M_\Xi(j)$ does not vanish. Under this assumption, multivariate cardinal interpolation with box-splines was studied by the authors [BHR85]. Only recently Chui, Diamond and Raphael [CDR87] consider the more general case when B_Ξ has isolated zeros or vanishes on a hypersurface. The above Corollary generalizes their results, establishing existence under no assumptions on B_Ξ whatsoever.

The simplest example which illustrates cardinal interpolation when the characteristic polynomial has zeros is quadratic spline interpolation at knots. In this case, the cardinal interpolant is a linear combination of the shifted B-splines $M_2(\cdot - 1/2 - j)$, $j \in \mathbb{Z}$, and the difference equation for the coefficients is

$$a(k)/2 + a(k-1)/2 = f(k).$$

This difference equation has the bounded fundamental solution

$$\check{b}(j) = \begin{cases} (-1)^j, & \text{for } j \geq 0; \\ (-1)^{j+1}, & \text{otherwise,} \end{cases}$$

showing that a bounded interpolant exists for functions f with $\sum_j |f(j)| < \infty$. This remains true in general:

Corollary 2. *For cardinal interpolation with shifted univariate B-splines, i.e., with* $b = M_r(\cdot - \tau)$, *there exists a bounded fundamental spline.*

The **proof** is a direct consequence of Theorem 2 and the fact that the Euler-Frobenius polynomial

$$B_{r,\tau}(x) := \sum_j \exp(ijx) M_r(j - \tau)$$

has at most one zero mod 2π and this zero is simple (see, e.g., [M76]). In fact, the Euler-Frobenius polynomial has a zero only when $\tau = 0$ $(1/2)$ and r is odd (even).

Finally, we discuss the approximation order of cardinal interpolation. Denote by

$$\mathcal{L}_h f := \sum_j L(\cdot/h - j) f(jh)$$

the cardinal interpolant with respect to the scaled lattice $h\mathbb{Z}^d$, $h \leq 1$. We show that $\mathcal{L}_h$ has the optimal approximation order as $h \to 0$.

Theorem 3. *Assume that* b *is* **normalized**, *i.e., that* $\sum_j b(j) = 1$, *and that any polynomial* p *of total degree* $< r$ *can be written as a linear combination of integer translates of* b. *Then, for any function* $f \in C^{m+d+r}$ *(where* m *is defined in (14)) with compact support* Ω,

$$\|f - \mathcal{L}_h f\|_{\infty,\Omega} = O(h^r).$$

This result is a generalization of Theorem 1 in [CDR87] in that here there are no assumptions made on the characteristic polynomial B. The asserted convergence rate is optimal in the following sense. According to wellknown results by Strang&Fix [SF69] and others, for a normalized b, the distance of all sufficiently smooth functions f from the space

$$S_h := \operatorname{span}\Big(b(\cdot/h - j)\Big)_{j \in \mathbb{Z}^d}$$

is $O(h^r)$ if and only if

$$\pi_{<r} \subset S_1, \tag{16}$$

i.e., all polynomials of degree $< r$ can be written as linear combinations of integer translates of b. In the proof of the theorem, we will make use of the fact detailed in [B87] that (16) is equivalent to having convolution with b **degree-preserving** on $\pi_{<r}$. By this we mean that, with β denoting the linear map of semi-discrete convolution with b, i.e.,

$$\beta f := \sum_{j \in \mathbb{Z}^d} b(\cdot - j) f(j),$$

(and b normalized), we have

$$\beta p \in p + \pi_{<\deg p} \quad \text{for all } p \in \pi_{<r}. \tag{17}$$

The assumption that f has compact support is convenient because of the polynomial growth of the fundamental function L. For an arbitrary smooth function f, Theorem 3 can be applied to $f\chi$ where χ is a smooth cut-off function and this yields convergence of cardinal interpolants on arbitrary bounded domains.

We begin the **proof** with the following lemma.

Lemma. *For any (univariate) polynomial p with $p(0) = 0$,*

$$\mathcal{L}p(\beta) = p(\beta).$$

For the **proof**, it is sufficient to show that $\mathcal{L}\beta = \beta$, i.e., that $L * b_| = b$. But this follows at once from the fact that

$$L * b_| = (b * \check{b}) * b_| = b * (\check{b} * b_|)_| = b.$$

This gives the identity $\mathcal{L} = 1 - (1-\beta)^s + \mathcal{L}(1-\beta)^s$, $s \in \mathbb{N}$, and so provides the useful error formula

$$f - \mathcal{L}f = (1-\beta)^s f - \mathcal{L}(1-\beta)^s f. \tag{18}$$

We make use of this error formula in the following way. Since b is **normalized**, i.e., $\sum_{j\in\mathbb{Z}^d} b(j) = 1$,

$$(1-\beta)f(k) = \sum_{j\in\mathbb{Z}^d} b(k-j)\big(f(k) - f(j)\big)$$

is a first-order difference, hence $(1-\beta)^s f(k)$ is an s-order difference of f, hence boundable in terms of $D^s f$. Thus the second term in (18) is of the order $(R_f + sR_b)^{m+d}\|D^s f\|$, with R_f the radius of a ball containing the support of f. As to the first term, we use (17) to conclude that $(1-\beta)^r \pi_{<r} = \{0\}$. This implies that, for $s \geq r$,

$$\begin{aligned}|(1-\beta)^s f(x)| &\leq \text{const}_{b,s} \inf_{p\in\pi_{<r}} \sup_{|y-x|<sR_b} |(f-p)(y)| \\ &\leq \text{const}\,(sR_b)^r \|D^r f\|.\end{aligned}$$

This shows that

$$\|f - \mathcal{L}f\| \leq \text{const}\,\|D^r f\| + \text{const}\,(R_f + \text{const})^{m+d}\|D^s f\|, \tag{19}$$

with the various constants independent of f.

Since

$$\mathcal{L}_h f := \sum_{j\in\mathbb{Z}^d} L(\cdot/h - j) f(jh) = \sigma_{-h}\mathcal{L}\sigma_h f$$

with $\sigma_h f := f(\cdot h)$, it follows that the error in this scaled interpolation scheme is $O(h^r) + O(h^{-m-d}h^s)$, hence $O(h^r)$ if f is smooth enough, i.e., if f has $m+d+r$ continuous derivatives.

As an application of Theorem 3, consider interpolation with the Zwart element, i.e. the box-spline corresponding to $\Xi = \{(1,0),(1,1),(0,1),(-1,1)\}$. In this case, the difference equation for the coefficients λ of the fundamental spline is

$$\frac{1}{2}\check{b}(\nu,\mu) + \frac{1}{8}\Big(\check{b}(\nu-1,\mu) + \check{b}(\nu+1,\mu) + \check{b}(\nu,\mu-1) + \check{b}(\nu,\mu+1)\Big) = \delta(\mu,\nu)$$

which is very similar to the discrete Laplace equation. Therefore, by slightly modifying the computations at the end of section 2, one can show that

$$\check{b}(\nu,\mu) = 8(-1)^{\nu+\mu+1}\lambda(\nu,\mu)$$

with λ defined in (12). In particular the coefficients $\check{b}$ grow logarithmically and thus Theorem 3 holds with $f \in C^{4+\alpha}$ for any $\alpha > 0$.

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