Some Elementary Integrals in k-Dimensional Euclidean Space

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ABSTRACT

This note derives the expression $\pi^{k/2}\, r^k/(k/2)!$ for the k-dimensional volume of a sphere $\mathfrak{S}_k(r)$ of radius r, and $k\, \pi^{k/2} \times r^{k-1}/(k/2)!$ for the surface area of $\mathfrak{S}_k(r)$; $\frac{1}{2}\, h\, \pi^{(k-1)/2}\, r^{k-1}/(k/2)!$ for the volume of a cone $\mathfrak{C}_k(r, h)$ of height h with base $\mathfrak{S}_{k-1}(r)$, and $(r^2 + h^2)^{1/2}\, r^{k-2}\, \pi^{(k-1)/2}/[(k-1)/2]!$ for the area of the curved surface of $\mathfrak{C}_k(r, h)$. Formulae are obtained for k-dimensional integrals through the volume of $\mathfrak{S}_k(r)$, general spherical polar coordinates, and for the volume of an arbitrary solid whose boundary is given in the form $r = g(\mathbf{u})$, where $\mathbf{u}$ is a unit vector expressed in such coordinates; and finally, for integrals through the volumes, both of an axial sector of $\mathfrak{S}_k(r)$ of vertical angle ψ, and of the cone $\mathfrak{C}_k(r, h)$.

Some Elementary Integrals in k-Dimensional Euclidean Space

John H. Halton

Computer Sciences Department
The University of Wisconsin
Madison, Wisconsin 53706, U.S.A.

We first consider a *sphere* of radius $r > 0$ in k-dimensional Euclidean space $\mathbb{R}^k$:

$$\mathbb{S}_k(r) = \{\mathbf{x} = [x_i]_{i=1}^k : \Sigma_{i=1}^k |x_i|^2 \leqslant r^2\}. \tag{1}$$

If its *volume* is denoted by $V_k(r)$ and we write v_k for $V_k(1)$, then we see that

$$V_k(r) = r^k V_k(1) = r^k v_k, \tag{2}$$

and

$$V_k(r) = \int_0^\pi V_{k-1}(r\sin\theta)\ (r\sin\theta)\ d\theta, \tag{3}$$

or

$$v_k = v_{k-1} \int_0^\pi \sin^k\theta\ d\theta. \tag{4}$$

If we write

$$J_k = \int_0^\pi \sin^k\theta\ d\theta, \tag{5}$$

then

$$J_0 = \pi \quad \text{and} \quad J_1 = 2, \tag{6}$$

and also

$$v_1 = 2 \quad \text{and} \quad v_2 = \pi, \tag{7}$$

so that, formally, since $v_1 = v_0 J_1$,

$$v_0 = 1. \tag{8}$$

By (4) and (5), we have, in general, that

$$v_k = v_{k-1} J_k. \tag{9}$$

Now, integrating (5) by parts, we see that, for $k \geqslant 2$,

$$J_k = \int_0^\pi \sin^{k-2}\theta \; (1 - \cos^2\theta) \; d\theta = J_{k-2} - \int_0^\pi \sin^{k-2}\theta \; \cos^2\theta \; d\theta$$

$$= J_{k-2} - \frac{1}{k-1} [\sin^{k-1}\theta \; \cos\theta]_0^\pi - \frac{1}{k-1} \int_0^\pi \sin^k\theta \; d\theta,$$

whence

$$J_k = \frac{k-1}{k} J_{k-2}. \tag{10}$$

It follows that

$$J_{2r} = \frac{(2r-1)(2r-3)\ldots(3)(1)}{(2r)\;\;(2r-2)\ldots(4)(2)} J_0 = \frac{(2r)!}{2^{2r}\;(r!)^2}\,\pi, \tag{11}$$

and

$$J_{2r-1} = \frac{(2r-2)(2r-4)\ldots(4)(2)}{(2r-1)(2r-3)\ldots(5)(3)} J_1 = \frac{2^{2r}\;(r!)^2}{(2r)!}\,\frac{1}{r}, \tag{12}$$

as is easily verified.

We proceed to apply (11) and (12) to (9), observing that

$$v_k = J_k \; J_{k-1} \; \cdots \; J_2 \; J_1 \; v_0; \tag{13}$$

so that

$$v_{2r} = (J_{2r} \; J_{2r-1})(J_{2r-2} \; J_{2r-3})\cdots(J_2 \; J_1) \; v_0$$

$$= \frac{\pi}{r}\,\frac{\pi}{r-1}\,\frac{\pi}{r-2} \cdots \frac{\pi}{1} = \frac{\pi^r}{r!}, \tag{14}$$

and similarly,

$$\begin{aligned} v_{2r-1} &= J_{2r-1}\,(J_{2r-2}\,J_{2r-3})(J_{2r-4}\,J_{2r-5})\cdots(J_2\,J_1)\;v_0 \\ &= \frac{2^{2r}\,(r!)^2}{(2r)!}\,\frac{1}{r}\,\frac{\pi}{r-1}\,\frac{\pi}{r-2}\cdots\frac{\pi}{1} \\ &= \frac{\pi^{r-1}\,2^{2r}\,r!}{(2r)!} = \frac{\pi^{r-1}}{(r-\frac{1}{2})(r-\frac{3}{2})\cdots(\frac{3}{2})(\frac{1}{2})}\,. \end{aligned} \tag{15}$$

We recall that the Gamma Function integral

$$\Gamma(z+1) = \int_0^\infty t^z\,e^{-t}\,dt \tag{16}$$

yields, by integration by parts, that, if $z > 0$,

$$\Gamma(z+1) = [-t^z\,e^{-t}]_0^\infty + z\int_0^\infty t^{z-1}\,e^{-t}\,dt = z\,\Gamma(z). \tag{17}$$

Thus, if z is an integer, we have that

$$\Gamma(z+1) = z!\;\Gamma(1) = z! \tag{18}$$

and we extend the definition of the factorial function by this identity. We now note that, if $r \geqslant 1$ is an integer,

$$(r-\tfrac{1}{2})! = (r-\tfrac{1}{2})(r-\tfrac{3}{2})\cdots(\tfrac{3}{2})(\tfrac{1}{2})\;(-\tfrac{1}{2})! \tag{19}$$

and

$$(-\tfrac{1}{2})! = \int_0^\infty t^{-1/2}\,e^{-t}\,dt = 2\int_0^\infty e^{-u^2}\,du = \sqrt{\pi}; \tag{20}$$

so that (15), (19), and (20) yield that $v_{2r-1} = \pi^{r-1}\sqrt{\pi}/(r-\frac{1}{2})!$ We now see that (14) and (15) take the same form: whether k be odd or even,

$$v_k = \frac{\pi^{k/2}}{(k/2)!}\,. \tag{21}$$

We now turn to the *surface area* $S_k(r)$ of $\mathfrak{S}_k(r)$, and write s_k for $S_k(1)$. We then see that

$$S_k(r) = r^{k-1} S_k(1) = r^{k-1} s_k. \tag{22}$$

Clearly,

$$V_k(r) = \int_0^r S_k(u)\, du, \tag{23}$$

whence

$$S_k(r) = \frac{d}{dr} V_k(r) = k\, r^{k-1} v_k, \tag{24}$$

by (2); and so, by (21) and (22),

$$s_k = k\, v_k = \frac{k\, \pi^{k/2}}{(k/2)!}\,. \tag{25}$$

In particular, we observe that

$$v_1 = 2,\quad v_2 = \pi,\quad v_3 = \frac{4}{3}\,;\qquad \text{and}\qquad s_1 = 2,\quad s_2 = 2\pi,\quad s_3 = 4\pi, \tag{26}$$

by direct application of (21) and (25), confirming well-known results.

Finally, we consider the *cone* $\mathfrak{C}_k(r, h)$, whose *base* is $\mathfrak{S}_{k-1}(r)$ and whose *height* is h. The *volume* of this solid in $\mathfrak{R}^k$ is clearly

$$C_k(r, h) = \int_0^h V_{k-1}(\tfrac{r}{h}\, x)\, dx, \tag{27}$$

so that, by (2),

$$C_k(r, h) = (\tfrac{r}{h})^{k-1}\, v_{k-1} \int_0^h x^{k-1}\, dx \tag{28}$$

$$= \frac{h}{k}\, r^{k-1}\, v_{k-1}. \tag{29}$$

The *area of the curved surface* of $\mathfrak{C}_k(r, h)$ may be determined in two ways. First, we see that it is

$$D_k(r,\ h) = \int_0^h S_{k-1}(\frac{r}{h}\ x)\ \mathrm{d}x\ \frac{(r^2 + h^2)^{1/2}}{h} \tag{30}$$

$$= (\frac{r}{h})^{k-2}\ s_{k-1} \int_0^h x^{k-2}\ \mathrm{d}x\ \frac{(r^2 + h^2)^{1/2}}{h} \tag{31}$$

$$= \frac{(r^2 + h^2)^{1/2}}{k - 1}\ r^{k-2}\ s_{k-1}, \tag{32}$$

by (22). On the other hand, we see that

$$C_k(r,\ h) = \int_0^h D_k(\frac{r}{h}\ x,\ x)\ \mathrm{d}x\ \frac{r}{(r^2 + h^2)^{1/2}}\ ; \tag{33}$$

so that

$$D_k(r,\ h) = \frac{\mathrm{d}}{\mathrm{d}y}\ C_k(\frac{r}{h}\ y,\ y)\ \frac{(r^2 + h^2)^{1/2}}{r}\ \Bigg|_{\ y=h} \tag{34}$$

$$= \frac{\mathrm{d}}{\mathrm{d}y}\ \frac{y}{k}\ (\frac{r}{h}\ y)^{k-1}\ v_{k-1}\ \frac{(r^2 + h^2)^{1/2}}{r}\ \Bigg|_{\ y=h}$$

$$= (r^2 + h^2)^{1/2}\ r^{k-2}\ v_{k-1}. \tag{35}$$

We now see by (25) that (32) and (35) agree.

By (21) and (25), we finally obtain that, if $r/h = \tan\psi$ and

$$C_k(r,\ h) = r^k\ c_k(\psi) \quad \text{and} \quad D_k(r,\ h) = r^{k-1}\ d_k(\psi), \tag{36}$$

then

$$c_k(\psi) = \frac{\pi^{(k-1)/2}}{(\frac{k-1}{2})!}\ \frac{\cot\psi}{k} \quad \text{and} \quad d_k(\psi) = \frac{\pi^{(k-1)/2}}{(\frac{k-1}{2})!}\ \mathrm{cosec}\psi. \tag{37}$$

In particular,

$$c_2(\psi) = \cot\psi,\ c_3(\psi) = \frac{\pi}{3}\cot\psi, \quad \text{and} \quad d_2(\psi) = 2\ \mathrm{cosec}\psi,\ d_3(\psi) = \pi\ \mathrm{cosec}\psi, \tag{38}$$

confirming well-known results.

and then (41) holds, as required; and indeed

$$\|\mathbf{x}\| = (\Sigma_{i=1}^{k} |x_i|^2)^{1/2} = r. \tag{43}$$

We now note that

$$\frac{\partial x_i}{\partial r} = u_i, \text{ and } \frac{\partial x_i}{\partial \theta_{j-1}} = \begin{cases} 0 & \text{if } j > i+1 \\ -ru_i \tan\theta_{j-1} & \text{if } j = i+1 \\ ru_i \cot\theta_{j-1} & \text{if } j < i+1 \end{cases}; \tag{44}$$

so that the *Jacobian* of the transformation is

$$\mathfrak{J}_k = \partial(x_1, x_2, \ldots, x_k)/\partial(r, \theta_1, \theta_2, \ldots, \theta_{k-1}) \tag{45}$$

$$= \begin{vmatrix} u_1 & -ru_1 \tan\theta_1 & 0 & \cdots & 0 & 0 \\ u_2 & ru_2 \cot\theta_1 & -ru_2 \tan\theta_2 & \cdots & 0 & 0 \\ u_3 & ru_3 \cot\theta_1 & ru_3 \cot\theta_2 & \cdots & 0 & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots \\ u_{k-2} & ru_{k-2} \cot\theta_1 & ru_{k-2} \cot\theta_2 & \cdots & -ru_{k-2} \tan\theta_{k-2} & 0 \\ u_{k-1} & ru_{k-1} \cot\theta_1 & ru_{k-1} \cot\theta_2 & \cdots & ru_{k-1} \cot\theta_{k-2} & -ru_{k-1} \tan\theta_{k-1} \\ u_k & ru_k \cot\theta_1 & ru_k \cot\theta_2 & \cdots & ru_k \cot\theta_{k-2} & ru_k \cot\theta_{k-1} \end{vmatrix};$$

$$= r^{k-1} u_1 u_2 \cdots u_k \Delta_k; \tag{46}$$

where

$$\Delta_k = \begin{vmatrix} 1 & -\tan\theta_1 & 0 & \cdots & 0 & 0 \\ 1 & \cot\theta_1 & -\tan\theta_2 & \cdots & 0 & 0 \\ 1 & \cot\theta_1 & \cot\theta_2 & \cdots & 0 & 0 \\ & \cdots & \cdots & & \cdots & \\ 1 & \cot\theta_1 & \cot\theta_2 & \cdots & -\tan\theta_{k-2} & 0 \\ 1 & \cot\theta_1 & \cot\theta_2 & \cdots & \cot\theta_{k-2} & -\tan\theta_{k-1} \\ 1 & \cot\theta_1 & \cot\theta_2 & \cdots & \cot\theta_{k-2} & \cot\theta_{k-1} \end{vmatrix} = \frac{\Delta_{k-1}}{\sin\theta_{k-1} \cos\theta_{k-1}}, \tag{47}$$

We now turn to the general question of integrating a function $f(\mathbf{x}) = f(x_1, x_2, \ldots, x_k)$ through the volume of the sphere $\mathfrak{S}_k(R)$ of radius R:

$$I = \int_{\mathfrak{S}_k(R)} dV_k(\mathbf{x})\ f(\mathbf{x}) = \underbrace{\int dx_1 \int dx_2 \ \cdots \int dx_k}_{\mathbf{x} \in \mathfrak{S}_k(R)} f(x_1, x_2, \ldots, x_k), \quad (39)$$

where $dV_k(\mathbf{x})$ denotes an element of volume in $\mathfrak{S}_k(R)$ [with $\mathbf{x} \in \mathfrak{S}_k(R)$ understood.] Integrating in *spherical shells*, as in (23), we get that

$$I = \int_0^R dr \int_{\partial\mathfrak{S}_k(r)} dS_k(r\,\mathbf{u})\ f(r\,\mathbf{u}) = \int_0^R dr\ r^{k-1} \int_{\partial\mathfrak{S}_k(1)} dS_k(\mathbf{u})\ f(r\,\mathbf{u}), \quad (40)$$

where $\partial\mathfrak{S}_k(r)$ denotes the *surface* of the sphere $\mathfrak{S}_k(r)$, $\mathbf{x} = r\,\mathbf{u}$ is a point on this surface, $\mathbf{u}$ denotes a *unit vector*, with

$$\|\mathbf{u}\| = (\Sigma_{i=1}^k\ |u_i|^2)^{1/2} = 1, \quad (41)$$

$dS_k(r\,\mathbf{u})$ and $dS_k(\mathbf{u})$ denote element of surface area on $\partial\mathfrak{S}_k(r)$ and $\partial\mathfrak{S}_k(1)$, respectively, and we use (22) to scale back to the unit sphere $\mathfrak{S}_k(1)$. We may now explicitly express the point $\mathbf{x} = r\,\mathbf{u}$ on $\mathfrak{S}_k(r)$ in *spherical polar coordinates*:

$$\left.\begin{aligned}
x_1 &= r\,u_1 = r\cos\theta_1,\\
x_2 &= r\,u_2 = r\sin\theta_1\cos\theta_2,\\
x_3 &= r\,u_3 = r\sin\theta_1\sin\theta_2\cos\theta_3,\\
&\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots\\
x_{k-2} &= r\,u_{k-2} = r\sin\theta_1\sin\theta_2\cdots\sin\theta_{k-3}\cos\theta_{k-2},\\
x_{k-1} &= r\,u_{k-1} = r\sin\theta_1\sin\theta_2\cdots\sin\theta_{k-3}\sin\theta_{k-2}\cos\theta_{k-1},\\
x_k &= r\,u_k = r\sin\theta_1\sin\theta_2\cdots\sin\theta_{k-3}\sin\theta_{k-2}\sin\theta_{k-1};
\end{aligned}\right\} \quad (42)$$

as is readily verified by expanding the determinant Δ_k by its last column (and noting that the last two rows are identical, except in the last column, and that Δ_{k-1} is the leading principal (k-1)-rowed minor of Δ_k.) We may now combine (42), (46), and (47), to yield that

$$\mathfrak{J}_k = r^{k-1} \sin^{k-2}\theta_1 \sin^{k-3}\theta_2 \cdots \sin^2\theta_{k-3} \sin\theta_{k-2}. \tag{48}$$

Returning to (40), we now see that

$$I = \int_0^R dr\, r^{k-1} \int_0^\pi d\theta_1 \sin^{k-2}\theta_1 \int_0^\pi d\theta_2 \sin^{k-3}\theta_2 \cdots \int_0^\pi d\theta_{k-3} \sin^2\theta_{k-3} \int_0^\pi d\theta_{k-2} \sin\theta_{k-2} \int_0^{2\pi} d\theta_{k-1}\, f(r\mathbf{u}). \tag{49}$$

Incidentally, we see that we obtain an independent derivation of (13), by putting $f(\mathbf{x}) = 1$ in (49); since this allows us to separate the integrals in (49) to give that

$$v_k = \frac{1}{k} J_{k-2} J_{k-3} \cdots J_2 J_1 2\pi = \frac{1}{k} \frac{\pi^{(k-2)/2}}{(\frac{k-2}{2})!} 2 = \frac{\pi^{k/2}}{(k/2)!}$$

$$= J_k J_{k-1} \cdots J_2 J_1 v_0, \text{ since } v_0 = 1. \tag{50}$$

[We use the earlier-proven fact that $J_1 J_2 \cdots J_{k-1} J_k = \pi^{k/2}/(k/2)!$]

Similarly, if we seek to determine the volume of a solid given by an equation of the (polar) form

$$r = g(\mathbf{u}), \tag{51}$$

we see that this will be the integral

$$Q_k = \int_0^{\pi} d\theta_1 \sin^{k-2}\theta_1 \cdots \int_0^{\pi} d\theta_{k-2} \sin\theta_{k-2} \int_0^{2\pi} d\theta_{k-1} \frac{[g(\mathbf{u})]^k}{k}, \qquad (52)$$

where we have integrated with respect to r from 0 to $g(\mathbf{u})$.

If we wish to integrate only over an *axial sector* of $\mathfrak{S}_k(R)$ of *vertical angle* ψ, we may take the axis of the sector to be the axis of the coordinate x_1, i.e., the axis of the angle θ_1. Then the integral of $f(\mathbf{x})$ becomes [compare (49)]

$$I(\psi) = \int_0^R dr\, r^{k-1} \int_0^{\psi} d\theta_1 \sin^{k-2}\theta_1 \int_0^{\pi} d\theta_2 \sin^{k-3}\theta_2 \cdots \int_0^{\pi} d\theta_{k-3} \sin^2\theta_{k-3} \int_0^{\pi} d\theta_{k-2} \sin\theta_{k-2} \int_0^{2\pi} d\theta_{k-1} f(r\,\mathbf{u}). \qquad (53)$$

Finally, we turn to the cone $\mathfrak{C}_k(R, H)$ with $R/H = \tan\psi$, so that the vertical angle of the cone is ψ. We may again adopt the spherical polar coordinates (42), and then we see that, if the axis of the cone is the axis of x_1 (i.e. of θ_1), then $(r\sin\theta_1, \theta_2, \theta_3, \ldots, \theta_{k-1})$ are spherical polar coordinates in the $(k-1)$-dimensional base $\mathfrak{S}_{k-1}(R)$ of $\mathfrak{C}_k(R, H)$. So the integral of $f(\mathbf{x})$ over $\mathfrak{C}_k(R, H)$ becomes

$$K = \int_{\mathfrak{C}_k(R,\ H)} dC_k(R, H)\ f(\mathbf{x}) = \int_0^H dh \int_0^{Rh/H} dt\ t^{k-2} \int_0^{\pi} d\theta_2 \sin^{k-3}\theta_2 \cdots \int_0^{\pi} d\theta_{k-2} \sin\theta_{k-2} \int_0^{2\pi} d\theta_{k-1} f(\mathbf{x}), \qquad (54)$$

where we put $h = r\cos\theta_1$ and $t = r\sin\theta_1$. We may now compute the volume of the cone, just like that of the sphere, by putting $f(\mathbf{x}) = 1$. The integrals then separate, and we obtain that, by (50).

$$C_k(R, H) = J_{k-3}\, J_{k-4} \cdots J_2\, J_1\, 2\pi \int_0^H dh \int_0^{Rh/H} dt\, t^{k-2}$$

$$= (k - 1)\, v_{k-1} \int_0^H dh\, \frac{1}{k-1} \left(\frac{Rh}{H}\right)^{k-1}$$

$$= v_{k-1} \left(\frac{R}{H}\right)^{k-1} \frac{H^k}{k} = \frac{H}{k} R^{k-1}, \qquad (55)$$

in agreement with (29).

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