GENERALIZED LINEAR COMPLEMENTARITY PROBLEMS AS LINEAR PROGRAMS

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ABSTRACT

A generalized linear complementarity problem which is equivalent to finding a root of a piecewise-linear system of equations is shown to be solvable if and only if a related linear programming problem is solvable. Furthermore each solution of the linear programming problem solves the generalized linear complementarity problem and is a root of the equivalent piecewise-linear system of equations.

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This work contains an extension of the essential idea underlying the approach recently proposed by the author [9,10,11,12] to the generalized linear complementarity problem of finding an x in the n-dimensional real Euclidean space R^n such that

$$M^i x + q^i \geqq 0, \quad \prod_{i=1}^{k} (M^i x + q^i)_j = 0, \quad j=1,\ldots\ldots,m \tag{1}$$

where $\prod$ is the product symbol, subscripts denote vector elements, k is a given integer bigger than 1, and for $i=1,\ldots\ldots,k$, M^i are given $m\times n$ real matrices and q^i are given vectors in R^m . This problem subsumes the fundamental linear complementarity problem [2]

$$w = Mx+q \geqq 0, \quad x \geqq 0, \quad x^T w = 0 \tag{2}$$

which is obtained from (1) by setting $k=2$, $m=n$, $M^1 = M$, $M^2 = I$, I being the identity matrix, $q^1 = q$ and $q^2 = 0$. Problem (1) also includes the generalized linear complementarity problem of Cottle and Dantzig [3] as a special case. In addition problem (1) is equivalent to the problem of finding an x in R^n which satisfies the following piecewise-linear system of equations

$$\underset{1\leqq i\leqq k}{\text{Minimum}}\{(M^i x+q^i)_j\} = 0, \quad j=1,\ldots,m \tag{3}$$

That (1) and (3) are equivalent follows from the following obvious equivalence for the real numbers $\alpha^1,\ldots\ldots,\alpha^k$

$$\min_{1\leqq i\leqq k} \{\alpha^i\} = 0 \Leftrightarrow \prod_{i=1}^{k} \alpha^i = 0, \; \alpha^i \geqq 0, \; i=1,\ldots,k \tag{4}$$

The piecewise-linear system (3) which has been considered by Eaves [6] occurs in certain nonlinear networks [7,8,1] and is unsolved in its general form except by enumerative methods. Our approach here, which is a generalization of the approach of [9,10,11,12] for the linear complementarity problem (2), is to look for a vector p in R^n such that each solution of the linear program

$$\text{Minimize } p^T x \quad \text{subject to } M^i x + q^i \geqq 0, \quad i=1,\ldots.,k \tag{5}$$

is also a solution of the generalized linear complementarity problem (1). Our principal result, Theorem 1, states that the generalized linear complementarity problem (1) or equivalently the piecewise linear system (3) is solvable if and only if the linear program (5) is solvable for some p satisfying conditions (8) and (9), and that each solution of the linear program solves the generalized linear complementarity problem (1) and equivalently the piecewise-linear system (3). In a related series of papers Cottle and Pang [4,5] and Pang [14] have shown, among other things, that the approach proposed by the author for the linear complementarity problem (2) has the interesting feature of finding the least element of a polyhedral set.

To establish our result, which is a generalization of [12, Theorem 3] we need the dual to the linear program (5), namely the linear program

$$\text{Maximize } \sum_{i=1}^{k} -(q^i)^T y^i \quad \text{subject to } \sum_{i=1}^{k} (M^i)^T y^i = p,\ y^i \geqq 0 \quad i=1,\ldots,k \tag{6}$$

where $y^i \in R^m$, for $i=1,\ldots,k$. Let the primal and dual feasible regions of (5) and (6) be denoted by S and T respectively, that is

$$\begin{aligned} S &= \{x \mid M^i x + q^i \geqq 0,\ i=1,\ldots,k,\ x \in R^n\} \\ T &= \{(y^1,\ldots,y^k) \mid \sum_{i=1}^{k} (M^i)^T y^i = p,\ y^i \geqq 0,\ y^i \in R^m,\ i=1,\ldots,k\} \end{aligned} \tag{7}$$

When S is nonempty, a necessary and sufficient condition that the dual linear programs (5) and (6) be solvable is that T be nonempty. This condition is equivalent to requiring that

$$p = \sum_{i=1}^{k} (M^i)^T s^i, \quad s^i \geqq 0, \quad s^i \in R^m, \quad i=1,\ldots,k \tag{8}$$

for some $s^i \in R^m$, $i=1,\ldots,k$. Throughout this work we shall assume that (8) holds and consequently can take (8) as the expression defining p. Let Z be the class of $m \times m$ real matrices with nonpositive off-diagonal elements. We are ready now to state and prove our principal result.

Theorem 1. The generalized linear complementarity problem (1) or equivalently the piecewise-linear system (3) has a solution if and only if the linear program (5) is solvable for some p defined by (8) with $(s^1,\ldots\ldots,s^k)$ in R^{km} satisfying the conditions

$$\begin{aligned}
&(a)\quad M^iQ = Z^i + q^ic^T, \quad i=1,\ldots,k \\
&(b)\quad \sum_{i=1}^{k} (s^i)^T Z^i \geqq 0 \\
&(c)\quad \sum_{i=1}^{k} (s^i)^T Z^i + c^T > 0 \\
&(d)\quad \sum_{i=1}^{k} (s^i) > 0 \\
&\qquad c,\ s^1,\ldots,s^k \geqq 0, \quad Z^1,\ldots,Z^k \in Z, \quad Q\text{: unrestricted}
\end{aligned} \tag{9}$$

for some vector c in R^m, some $n\times m$ matrix Q and some $m\times m$ matrices $Z^1,\ldots,Z^k$. Furthermore each solution of the linear program (5) solves the generalized linear complementarity problem (1) or equivalently the piecewise-linear system (3).

Proof. (Necessity) Let x solve the generalized linear complementarity problem (1). Define s^i in R^m for $i=1,\ldots\ldots,k$ as follows

$$s_j^i = \begin{cases} 1 & \text{if } (M^ix+q^i)_j = 0 \\ 0 & \text{if } (M^ix+q^i)_j > 0 \end{cases} \qquad j=1,\ldots,m \tag{10}$$

It follows that $(s^i)^T(M^ix+q^i) = 0$ for $i=1,\ldots,k$ and consequently x solves the linear program (5) because $(s^1,\ldots\ldots,s^k)$ is dual feasible, that is it is in T, and

$$p^Tx = \sum_{i=1}^{k} (s^i)^T M^i x = \sum_{i=1}^{k} -(q^i)^T s^i$$

Furthermore conditions (9) are satisfied by $(s^1,\ldots,s^k)$ as defined in (10) above and $c=e$, $Z^i = -(M^ix+q^i)e^T$, $Q=-xe^T$, where e is a vector of ones in R^m .

(Sufficiency) Let conditions (8) and (9) hold and let x solve the linear program (5). We have that $(s^1,\ldots,s^k)$ is dual feasible. <u>If</u> $(s^1,\ldots,s^k)$ is also dual optimal, then

$$0 = \sum_{i=1}^{k} (q^i)^T s^i + \sum_{i=1}^{k} (s^i)^T M^i x = \sum_{i=1}^{k} (s^i)^T (M^i x + q^i) ,$$

and because $s^i \geqq 0$, $M^i x + q^i \geqq 0$ for $i=1,\ldots,k$, and $\sum_{i=1}^{k} s^i > 0$ it follows that $\prod_{i=1}^{k} (M^i x + q^i)_j = 0$, for $j=1,\ldots,m$, and consequently x solves the generalized linear complementarity problem (1). Suppose now that $(s^1,\ldots,s^k)$ is <u>not</u> dual optimal and that some $(y^1,\ldots,y^k)$ in R^{km} is dual optimal. Then

$$\sum_{i=1}^{k} (s^i)^T q^i + \sum_{i=1}^{k} (s^i)^T M^i x > 0$$

and hence we have from (9b), (9c) and the last inequality above that

$$\sum_{i=1}^{k} (s^i)^T Z^i + \sum_{i=1}^{k} (s^i)^T (M^i x + q^i) c^T > 0 \tag{11}$$

Let $Z^i = D - V^i$, for $i=1,\ldots,k$, where $D, V^1,\ldots,V^k$ are $m \times m$ nonnegative matrices and D is in addition a diagonal matrix with a positive diagonal. Then

$$\sum_{i=1}^{k} (s^i)^T Z^i = \sum_{i=1}^{k} (s^i)^T (M^i Q - q^i c^T) \qquad \text{(By (9a))}$$

$$= \sum_{i=1}^{k} (s^i)^T (M^i Q - q^i c^T) + \sum_{i=1}^{k} (y^i)^T (-M^i Q + D - V^i + q^i c^T) \qquad \text{(By (9a))}$$

$$= \sum_{i=1}^{k} -(s^i)^T q^i c^T + \sum_{i=1}^{k} (y^i)^T (D - V^i + q^i c^T) \qquad \text{(Because } (y^1,\ldots,y^k) \in T)$$

$$\leqq \sum_{i=1}^{k} -(s^i)^T q^i c^T + \sum_{i=1}^{k} (y^i)^T (D + q^i c^T) \qquad \text{(Because } y^i \geqq 0,\ V^i \geqq 0 \text{ for } i=1,\ldots,k)$$

$$= -\sum_{i=1}^{k} (s^i)^T(M^i x+q^i)c^T + \sum_{i=1}^{k} (y^i)^T D$$

(Because $(y^1,\ldots,y^k)$ is dual optimal)

Therefore

$$\sum_{i=1}^{k} (y^i)^T D \geqq \sum_{i=1}^{k} (s^i)^T Z^i + \sum_{i=1}^{k} (s^i)^T(M^i x+q^i)c^T \tag{12}$$

By using (11) in (12) and recalling that D is a diagonal matrix with a positive diagonal we obtain that

$$\sum_{i=1}^{k} y^i > 0 \tag{13}$$

But because x and $(y^1,\ldots,y^k)$ solve the dual linear programs (5) and (6) we have that

$$\sum_{i=1}^{k} (y^i)^T(M^i x+q^i) = 0 \tag{14}$$

From (13) and (14) we have upon noting that $y^i \geqq 0$, $M^i x+q^i \geqq 0$ for $i=1,\ldots,k$, that $\prod_{i=1}^{k} (M^i x+q^i)_j = 0$, for $j=1,\ldots,m$. Hence again x solves the generalized linear complementarity (1). □

We note that the choice of $s^1,\ldots,s^k,c$, $Z^1,\ldots\ldots,Z^k$ and Q in the necessity proof of Theorem 1 is a very particular choice and is by no means the only one that satisfies (9). Thus for example as shown in the proof of Corollary 1 below a different choice which does not require knowledge of a solution to the generalized linear complementarity problem (1) can satisfy the conditions (9). Also, in Examples 1 and 2 below we choose for $s^1,\ldots,s^k$, c, $Z^1,\ldots\ldots,Z^k$ and Q, values other than those given in the necessity proof of Theorem 1.

<u>Corollary 1</u>. Let $m=n$, $M^i \epsilon Z$, for $i=1,\ldots\ldots,k$, let S be nonempty and let $\sum_{i=1}^{k} (s^i)^T M^i > 0$ for some nonnegative $(s^1,\ldots\ldots,s^k)$ in R^{kn}. Then the generalized linear complementarity problem (1) or equivalently the piecewise-linear system (3) has a solution which

can be obtained as a solution of the solvable linear program (5) with $p = \sum_{i=1}^{k} (M^i)^T s^i$.

Proof. This corollary follows from Theorem 1 by noting that conditions (9a), (9b) and (9c) are satisfied by taking $c=0$, $Q=I$ and $Z^i = M^i$ for $i=1,\ldots,k$. If condition (9d) is not satisfied we would have $s^i_j = 0$ for $i=1,\ldots,k$ and some $j, 1\leq j\leq m$, which when combined with (9c) gives the contradiction

$$0 < \sum_{i=1}^{k} ((s^i)^T Z^i)_j \leq \sum_{i=1}^{k} s^i_j Z^i_{jj} = 0$$

Hence all the conditions of (9) are satisfied, the linear program (5) is solvable and any solution of it solves (1) or equivalently (3). □

The following example due to Eaves [6] illustrates the use of Theorem 1.

Example 1 [6, p. 100]

$$M^1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}, \quad q^1 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad M^2 = \begin{pmatrix} 2 & -1 \\ -2 & -1 \end{pmatrix}, \quad q^2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$Z^1 = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad Z^2 = I, \qquad Q = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} \end{pmatrix}, \qquad c = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \end{pmatrix}$$

$$s^1 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \qquad s^2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad p = \begin{pmatrix} 2 \\ -4 \end{pmatrix} .$$

GLCP: $\text{Min}\{-x_1-x_2+2,\ 2x_1-x_2-1\} = 0$

$\text{Min}\{x_1-x_2+2,\ -2x_1-x_2-1\} = 0$

LP: $\text{Min} \quad 2x_1 - 4x_2$

subject to
$$\begin{aligned} -x_1 - x_2 + 2 &\geqq 0 \\ x_1 - x_2 + 2 &\geqq 0 \\ 2x_1 - x_2 - 1 &\geqq 0 \\ -2x_1 - x_2 - 1 &\geqq 0 \end{aligned}$$

The point $x_1 = 0$, $x_2 = -1$ with $M_1^2 x + q_1^2 = 0$, $M_2^2 x + q_2^2 = 0$, is a solution to both the GLCP and LP.

The following example is an application of Corollary 1.

<u>Example 2</u>

$$M^1 = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \quad M^2 = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \quad M^3 = \begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix}$$

$$q^1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad q^2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad q^3 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$s^1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad s^2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad s^3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad p = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

GLCP: $\text{Min}\{x_1 - x_2 + 1,\ x_1 - x_2 + 2,\ -x_1 - x_2 + 1\} = 0$

$\text{Min}\{-x_2 - 1,\ 2x_2 + 3,\ -x_1 - 1\} = 0$

LP:
$$\begin{aligned} \text{Min}\quad & x_1 + x_2 \\ \text{subject to}\quad & x_1 - x_2 + 1 \geqq 0 \\ & -x_2 - 1 \geqq 0 \\ & x_1 - x_2 + 2 \geqq 0 \\ & 2x_2 + 3 \geqq 0 \\ & -x_1 - x_2 + 1 \geqq 0 \\ & -x_1 - 1 \geqq 0 \end{aligned}$$

The point $x_1 = -\frac{5}{2}$, $x_2 = -\frac{3}{2}$ with $M_1^1 x + q_1^1 = 0$ and $M_2^2 x + q_2^2 = 0$, solves both the GLCP and LP.

Conditions (8) and (9) so far are probably the simplest characterization of generalized linear complementarity problems as linear programs. Many cases are given in [11] for which the conditions (8) and (9) are satisfied for the simpler linear complementarity problem (2). What is still unknown is the scope of these conditions, that is the class of linear complementarity problems and generalized linear complementarity problems for which conditions (8) and (9) can be "easily" verified. It is hoped that future research will clarify and enlarge this scope.

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