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THE METHOD OF CHRISTOPHERSON FOR SOLVING FREE BOUNDARY PROBLEMS FOR INFINITE JOURNAL BEARINGS BY MEANS OF FINITE DIFFERENCES

by

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1. Introduction

A journal bearing consists of a rotating cylinder which is separated from a "bearing surface" by a thin film of lubricating fluid (see Figure 1). The fluid is fed in at A and flows out at B. The width of the film is smallest at C, and we set $t = \theta/\theta_C$ where θ is as shown in Figure 1.

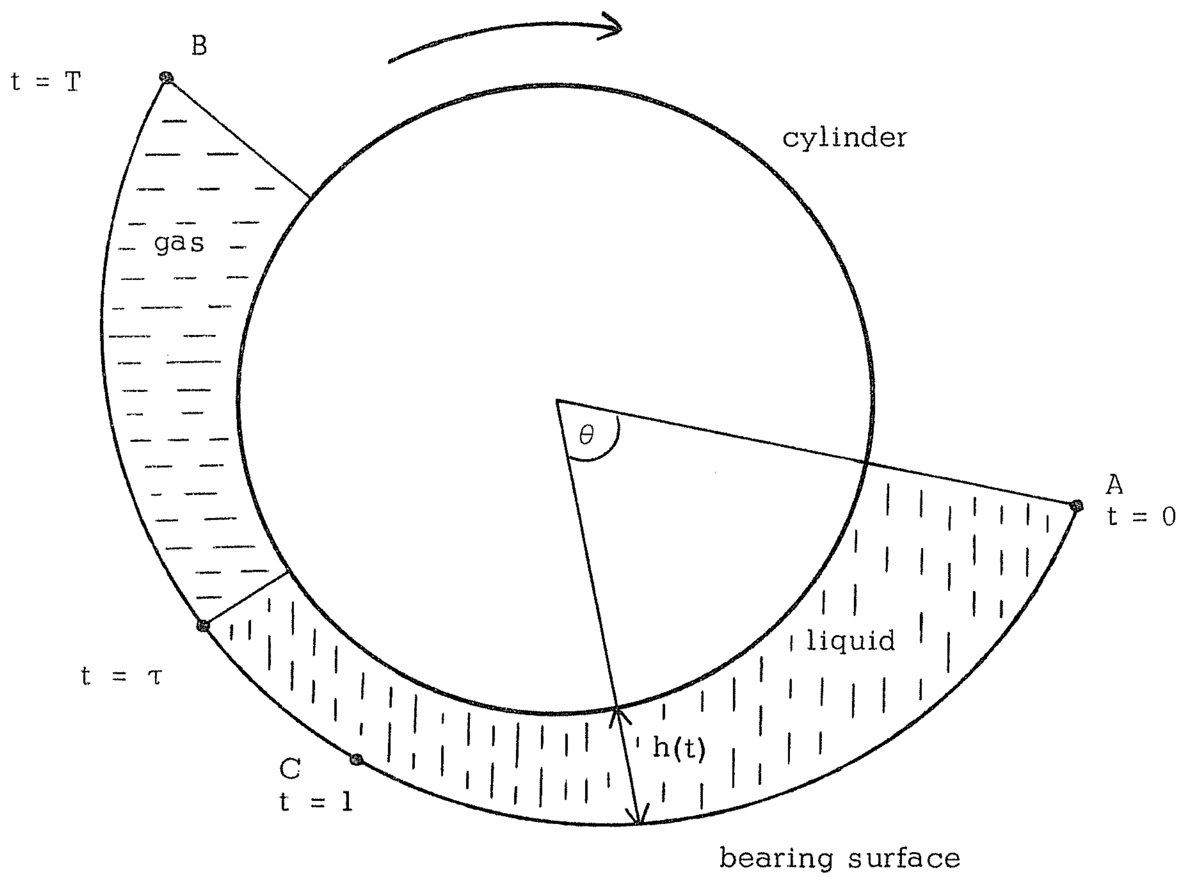

Figure 1

Cross-section of a journal bearing.

Between C and B the width of the film increases so that the pressure in the lubricating fluid may be expected to decrease. We assume that for $t = \tau$ the pressure becomes so low that the fluid vaporizes. The point $t = \tau$, the interface between the two phases of the fluid, is called the free boundary.

The mathematical problem can now be formulated (see Pinkus and Sternlicht [9, p. 41 and p. 46]):

Problem 1

Find a function $p(t)$ and a constant τ such that $p \in \mathcal{C}[0, T] \cap \mathcal{C}^{(2)}(0, \tau)$, and

$$\mathcal{L}\, p(t) = \frac{d}{dt}\left[h^3(t)\frac{dp}{dt}\right] - \frac{dh}{dt} = 0, \quad 0 < t < \tau, \tag{1.1}$$

$$p(t) = 0, \quad \tau \le t \le T, \tag{1.2}$$

$$p(0) = 0, \tag{1.3}$$

$$\frac{d}{dt}\, p(\tau) = 0. \tag{1.4}$$

In Problem 1, $p(t)$ is proportional to the fluid pressure, while equation (1.1) is Reynolds' equation for the pressure in a lubricating film.

It should be pointed out that although the above formulation is widely used, there has been considerable discussion as to whether it is an accurate model of the physical problem. Firstly, many assumptions are made in deriving Reynolds' equation (1.1) (Pinkus and Sternlicht [9, p. 6]); for a discussion of the validity of these assumptions see Halton [6] and Hersey [7]. Secondly, conditions other than (1.2) through (1.4) are sometimes used; for a discussion of this see Birkhoff and Hays [1].

In order that Problem 1 be well-defined, it is necessary that h(t), the width of the film, satisfy certain conditions. Throughout this paper we will assume that $h \in \mathcal{C}^{(1)}[0, T]$ and that

$$h(t) > 0, \quad t \in [0, T], \tag{1.5}$$

$$\left.\begin{aligned} \frac{dh}{dt} &< 0, \quad t \in (0, 1), \\ \frac{dh}{dt} &> 0, \quad t \in (1, T), \end{aligned}\right\} \tag{1.6}$$

$$h(T) \geq h(0). \tag{1.7}$$

It will be shown in section 2 that conditions (1.5) through (1.7) ensure that there exists a unique solution to Problem 1.

Conditions (1.5) and (1.6) are always satisfied in practice, but this is not true of (1.7). However, as we shall see in section 2, condition (1.7) can be imposed without any loss of generality.

In 1941, Christopherson [3] proposed a method for solving journal bearing problems numerically. A partial analysis of the method was given by Gnanadoss and Osborne [5]. In the present paper we present a detailed analysis of Christopherson's method as applied to Problem 1.

The method of Christopherson consists of two steps. In the first step a discrete approximation to Problem 1 is set up; this discrete approximation is analysed in sections 3, 4, and 5. In the second step of Christopherson's method, the solution of the discrete approximation is computed by an iterative procedure; the convergence of this iterative procedure is studied in section 6. Finally, some numerical results are presented in section 7.

Acknowledgement

The author is indebted to T. Ladner who wrote the program used to obtain the numerical results in section 7.

2. The Analytic Problem

In this section we first analyse Problem 1 and prove that this problem has a unique solution. We then formulate another problem, Problem 2, and prove that it is equivalent to Problem 1. The reason for introducing Problem 2 is that, unlike Problem 1, it has a natural discrete analog, as we show in section 3.

First, we define several functions which will be used subsequently:

$$\vartheta_2(t) = \int_0^t h^{-2}(s)\,ds, \qquad t \in [0, T], \tag{2.1}$$

$$\vartheta_3(t) = \int_0^t h^{-3}(s)\,ds, \qquad t \in [0, T], \tag{2.2}$$

$$\varphi(t) = \vartheta_2(t)/\vartheta_3(t), \qquad t \in (0, T], \tag{2.3}$$

$$\psi(t) = h(t)\,\vartheta_3(t) - \vartheta_2(t), \qquad t \in [0, T]. \tag{2.4}$$

It is easily verified that,

$$\varphi'(t) = \psi(t)/\{h^3(t)[\vartheta_3(t)]^2\}, \quad t \in (0, T], \tag{2.5}$$

$$\psi'(t) = h'(t)\,\vartheta_3(t), \qquad t \in [0, T], \tag{2.6}$$

and

$$\psi(t) = \int_0^t [h(t) - h(s)]\, h^{-3}(s)\,ds, \quad t \in [0, T]. \tag{2.7}$$

Lemma 2.1

This is a unique constant σ, $1 < \sigma < T$, such that

$$\left.\begin{aligned} &\psi(t) < 0, \quad \text{for} \quad t \in (0, \sigma), \\ &\psi(\sigma) = 0, \\ &\psi(t) > 0, \quad \text{for} \quad t \in (\sigma, T]. \end{aligned}\right\} \tag{2.8}$$

Proof: From (1.6) and (2.6) it follows that $\psi(t)$ is strictly monotone decreasing for $t \in [0, 1]$ and strictly monotone increasing for $t \in [1, T]$. From (1.7) and (2.7) we see that $\psi(T) > 0$. Since $\psi(0) = 0$, the lemma follows.

Lemma 2.2

If σ is as in Lemma 2.1, then $\varphi(t)$ is strictly monotone decreasing for $t \in (0, \sigma]$ and strictly monotone increasing for $t \in [\sigma, T]$.

Proof: Follows immediately from (2.5) and (2.8).

Definition 2.1

For $a \in (0, T]$ let $q(t;a)$ be the function such that

$$\mathcal{L}\, q(t;a) = 0, \qquad t \in (0, a), \tag{2.9}$$

$$q(0;a) = 0, \tag{2.10}$$

$$q(t;a) = 0, \qquad t \in [a, T], \tag{2.11}$$

where $\mathcal{L}$ is as in (1.1).

Lemma 2.3

$q(t;a)$ is unique and

$$q(t;a) = \begin{cases} \vartheta_2(t) - \varphi(a)\,\vartheta_3(t), & t \in [0,a] \\ 0, & t \in [a,T]. \end{cases} \tag{2.12}$$

Proof: The lemma follows by integrating (2.9) twice and using (2.10) and (2.11).

Theorem 2.4

There is exactly one solution $\{p(t), \tau\}$ of Problem 1. If σ is as in Lemma 2.1, then

$$\tau = \sigma, \tag{2.13}$$

and

$$p(t) = q(t;\sigma). \tag{2.14}$$

Proof: The proof is a straight-forward generalization of previous results (Birkhoff and Hays [1, p. 132], Pinkus and Sternlicht [9, p. 46], and Gnanadoss and Osborne [5]).

From Definition 2.1 and Lemma 2.3 we see that if $\{p(t), \tau\}$ is a solution of Problem 1 then

$$p(t) = q(t;\tau) = \vartheta_2(t) - \varphi(\tau)\,\vartheta_3(t), \quad t \in [0,\tau],$$

from which it follows that

$$p'(\tau-0) = \psi(\tau)/\{h^3(\tau)\,\vartheta_3(\tau)\} .$$

Noting (1.4) and (2.8), the theorem follows.

We now formulate

Problem 2

Find $\{p(t), \tau\}$ such that

$$\tau = \sup\{a \in (0,T]: \; q(t;a) \geq 0 \quad \text{for} \quad t \in [0,T]\}, \tag{2.15}$$

$$p(t) = q(t;\tau) . \tag{2.16}$$

That is, find the largest interval in which a non-negative solution of Reynolds' equation exists.

Problem 2 was first suggested by Gnanadoss and Osborne [5], and the next theorem is a generalization of their results:

Theorem 2.5

Problems 1 and 2 are equivalent.

Proof: Rewriting (2.12) we obtain

$$q(t;a) = \begin{cases} [\varphi(t) - \varphi(a)]\,\vartheta_3(t), & t \in (0,a], \\ 0, & t \in [a,T] . \end{cases}$$

The theorem follows from (2.15), Lemma 2.2, and Theorem 2.4.

We conclude this section with a discussion of condition (1.7). First, we note that it is the role of condition (1.7) to ensure that the lubricating fluid occurs in both the liquid and gaseous phases. If (1.7) is not satisfied then it is possible, for example if B is close to C (see Figure 1), for the fluid to occur only in the liquid phase.

Next, we note that (1.7) is used only in Lemma 2.1. Examination of the proof of Lemma 2.1 shows that we may replace (1.7) by the weaker condition $\psi(T) > 0$.

Finally, we note that there is no loss of generality in assuming (1.7). For suppose that $\tilde{h}(t)$ and $\tilde{T}$ are such that $\tilde{h} \in \mathcal{C}^{(1)}[0, \tilde{T}]$ and

$$\tilde{h}(t) > 0, \qquad t \in [0, \tilde{T}],$$

$$\frac{d\tilde{h}}{dt} < 0, \qquad t \in (0, 1),$$

$$\frac{d\tilde{h}}{dt} > 0, \qquad t \in (1, \tilde{T}),$$

but that $\tilde{h}(\tilde{T}) < \tilde{h}(0)$. Let $T > \tilde{T}$ and $h \in \mathcal{C}^{(1)}[0, T]$ be such that (1.5) through (1.7) are satisfied and $h(t) = \tilde{h}(t)$ for $t \in [0, \tilde{T}]$; clearly, h and T can be chosen in many ways. Let Problem 1 and Problem $\tilde{1}$ be the problems corresponding to h and $\tilde{h}$, respectively. Clearly, if Problem $\tilde{1}$ has a solution $\{\tilde{p}(t), \tilde{\tau}\}$ then Problem 1 has a solution $\{p(t), \tau\}$ where $\tau = \tilde{\tau}$ and

$$p(t) = \begin{cases} \tilde{p}(t), & t \in [0, \tilde{T}] , \\ 0 , & t \in (\tilde{T}, T] . \end{cases}$$

Hence to solve Problem $\tilde{1}$ we may first find the solution $\{p(t), \tau\}$ of Problem 1; by Theorem 2.4 this solution exists. If $\tau > \tilde{T}$ Problem $\tilde{1}$ has no solution. On the other hand, if $\tau \leq \tilde{T}$, then Problem $\tilde{1}$ has the solution $\{\tilde{p}(t), \tilde{\tau}\}$ where $\tilde{\tau} = \tau$ and $\tilde{p}(t) = p(t)$ for $t \in [0, \tilde{T}]$.

3. The Discrete Approximation

In this section we formulate and analyse a discrete approximation to Problem 2, Problem 2D. We prove that the solution of Problem 2D exists and obtain an analytic expression for the solution.

Before defining Problem 2D, it is necessary to develop some preliminary results.

Throughout this section we denote by $\{p(t), \tau\}$ the solution of Problems 1 and 2. We subdivide the interval $[0, T]$ into N subintervals, each of length Δt, so that $N = T/\Delta t$. We seek an integer m and an $(N+1)$-vector $\underline{P} = \{P_j\}$, $j = 0, 1, \ldots, N$, such that $m\Delta t \approx \tau$ and $P_j \approx p(j\Delta t)$.

We approximate the Reynolds' equation (1.1) by the finite difference equation

$$(L\,\underline{P})_i = \frac{1}{(\Delta t)^2}\,\Delta(h^3_{i-\frac{1}{2}}\,\nabla P_i) - \frac{1}{\Delta t}\,\Delta h_{i-\frac{1}{2}} = 0, \qquad 0 < i < N, \tag{3.1}$$

where Δ and ∇ denote the forward and backward difference operators respectively, $h_i = h(i\Delta t)$, and $h_{i-\frac{1}{2}} = h([i-\frac{1}{2}]\,\Delta t)$.

In order to avoid certain trivial possibilities, we assume that

$$\Delta t \le 2/3\,, \tag{3.2}$$

$$N \ge 3\,. \tag{3.3}$$

We also make an additional assumption about $h(t)$, namely that

$$h(T - (\Delta t)/2) \geq h((\Delta t)/2). \tag{3.4}$$

Assumption (3.4) is trivially satisfied by, if necessary, slightly increasing T and modifying the definition of $h(t)$ appropriately (see section 2 where a similar device is used).

We set

$$I_2(i) = \sum_{j=1}^{i} (h_{j-\frac{1}{2}})^{-2}, \qquad 0 \leq i \leq N, \tag{3.5}$$

$$I_3(i) = \sum_{j=1}^{i} (h_{j-\frac{1}{2}})^{-3}, \qquad 0 \leq i \leq N, \tag{3.6}$$

$$\Phi_i = I_2(i)/I_3(i), \qquad 1 \leq i \leq N, \tag{3.7}$$

$$\Psi_i = h_{i+\frac{1}{2}} I_3(i) - I_2(i), \qquad 0 \leq i \leq N-1. \tag{3.8}$$

In (3.5), (3.6), and elsewhere in this paper, we follow the convention that

$$\sum_{j=1}^{0} (h_{j-\frac{1}{2}})^{-2} = \sum_{j=1}^{0} (h_{j-\frac{1}{2}})^{-3} = 0.$$

It is easily verified that

$$\Delta \Phi_i = \Psi_i/\{(h_{i+\frac{1}{2}})^3 I_3(i) I_3(i+1)\}, \quad 1 \leq i \leq N-1, \tag{3.9}$$

$$\Delta \Psi_i = (\Delta h_{i+\frac{1}{2}}) I_3(i+1), \qquad 0 \leq i \leq N-2, \tag{3.10}$$

$$\Psi_i = \sum_{j=1}^{i} (h_{i+\frac{1}{2}} - h_{j-\frac{1}{2}})(h_{j-\frac{1}{2}})^{-3}, \quad 0 \leq i \leq N-1. \tag{3.11}$$

Noting the analogy between equations (1.1) and (3.1) and between equations (2.1) through (2.7) and equations (3.5) through (3.11), we are led to

Lemma 3.1

There is a unique integer n, $1 - 3(\Delta t)/2 \leq n\Delta t \leq T-2(\Delta t)$, such that

$$\left.\begin{aligned} &\Psi_i < 0, \quad \text{for } 1 \leq i < n, \\ &\Psi_n \leq 0, \\ &\Psi_i > 0, \quad \text{for } n < i \leq N-1. \end{aligned}\right\} \tag{3.12}$$

Proof: Let $n_1 = n_1(\Delta t)$ be the largest integer such that $(n_1 + \frac{1}{2})\Delta t \leq 1$; noting (3.2) we see that $n_1 \geq 1$.

From (1.6) and (3.10) it follows that Ψ_i is strictly monotone decreasing for $0 \leq i \leq n_1$ and strictly monotone increasing for $n_1 < i \leq N-1$. From (3.8) we have that $\Psi_0 = 0$. From (1.6), (3.3), (3.4), and (3.11), we see that

$$\Psi_{N-1} \geqq (h_{N-\frac{1}{2}} - h_{3/2})(h_{3/2})^{-3} > (h_{N-1/2} - h_{1/2})(h_{3/2})^{-3} > 0.$$

Combining these results, we see that (3.12) holds for some unique n, where $n_1 \leq n \leq N-2$. The lemma now follows.

Lemma 3.2

If n is as in Lemma 3.1, then Φ_i is strictly monotone decreasing for $1 \le i \le n$, and strictly monotone increasing for $n+1 \le i \le N$, while $\Phi_{n+1} \le \Phi_n$.

Proof: Follows from (3.9) and (3.12).

Definition 3.1

For any integer ℓ, $1 \le \ell \le N$, let $\underset{\sim}{Q}(\ell) = \{Q_i(\ell)\}$ be the $(N+1)$-vector such that

$$[L\underset{\sim}{Q}(\ell)]_i = 0, \qquad 1 \le i \le \ell - 1, \tag{3.13}$$

$$Q_0(\ell) = 0, \tag{3.14}$$

$$Q_i(\ell) = 0, \qquad \ell \le i \le N, \tag{3.15}$$

where L is as in (3.1).

Lemma 3.3

$\underset{\sim}{Q}(\ell)$ is unique and

$$Q_i(\ell) = \begin{cases} \Delta t\{I_2(i) - \Phi_\ell\, I_3(i)\}, & 0 \le i \le \ell, \\ 0, \quad \ell \le i \le N. \end{cases} \tag{3.16}$$

Proof: The lemma follows by summing (3.13) twice and using (3.14) and (3.15).

We can now formulate the discrete analog of Problem 2:

Problem 2D

Find $\{\underline{P}, m\}$ such that,

$$m = \max\{\ell :\ 1 \le \ell \le N;\ Q_i(\ell) \ge 0 \ \text{ for } \ 0 \le i \le N\}, \qquad (3.17)$$

$$\underline{P} = \underline{Q}(m)\ . \qquad (3.18)$$

Theorem 3.4

There is exactly one solution $\{\underline{P}, m\}$ of Problem 2D. If n is as in Lemma 3.1 then

$$m = n + 1\ , \qquad (3.19)$$

and

$$\underline{P} = \underline{Q}(m)\ . \qquad (3.20)$$

Proof: Rewriting (3.16) we obtain

$$Q_i(\ell) = \begin{cases} \Delta t[\Phi_i - \Phi_\ell]\ I_3(i), & 1 \le i \le \ell\ , \\ 0, & \ell \le i \le N\ . \end{cases} \qquad (3.21)$$

The theorem now follows from (3.17) and Lemma 3.2.

We conclude this section by pointing out that there is an obvious discrete version of Problem 1, namely

Problem 1D

Find $\{\underline{P}, m\}$ such that

$$\underline{P} = \underline{Q}(m), \tag{3.22}$$

and

$$\nabla P_m = 0 . \tag{3.23}$$

The reader may have wondered why we have not considered Problem 1D. The reason is that, in general, Problem 1D will have no solution. For if $\{\underline{P}, m\}$ is a solution of Problem 1D then it follows from (3.22), (3.23), and (3.16), that m must be such that

$$h_{m-\frac{1}{2}} - \Phi_m = 0 . \tag{3.24}$$

Only in exceptional circumstances will there be an <u>integer</u> m satisfying (3.24), so that, in general, Problem 1D does not have a solution.

4. Error analysis

In this section we obtain bounds for the difference between the solution $\{p(t), \tau\}$ of Problems 1 and 2, and the solution $\{\underline{P}, m\}$ of Problem 2D. We use the same notation as in section 3. In order to obtain satisfactory error estimates, we assume that

$$h \in \mathcal{C}^{(2)}[0, T] . \tag{4.1}$$

Theorem 4.1

There are positive constants $(\Delta t)_0$ and K such that if

$$\Delta t \le (\Delta t)_0 , \tag{4.2}$$

then

$$|\tau - m\Delta t| \le 5(\Delta t)/8, \tag{4.3}$$

and

$$|p(j\Delta t) - P_j| \le K(\Delta t)^2, \ 0 \le j \le N. \tag{4.4}$$

Proof: In the proof, $K_1, K_2, \ldots,$ and $(\Delta t)_1, (\Delta t)_2, \ldots,$ will denote fixed positive constants which depend only upon $h(t)$ and T.

Since I_2 and I_3 coincide with the approximations to ϑ_2 and ϑ_3 obtained using the repeated mid-point rule (Hildebrand [8, p. 154]) we have, noting (4.1), that

$$\left.\begin{aligned} &|(\Delta t) I_2(i) - \vartheta_2(i\Delta t)| \le K_1(\Delta t)^2 , \\ &|(\Delta t) I_3(i) - \vartheta_3(i\Delta t)| \le K_2(\Delta t)^2 , \\ &\qquad\qquad \text{for } 0 \le i \le N . \end{aligned}\right\} \tag{4.5}$$

Rearranging (3.8) ,

$$\Psi_i = h_{i+\frac{1}{2}} \left[I_3(i) + \tfrac{1}{2} (h_{i+\frac{1}{2}})^{-3} \right] - \left[I_2(i) + \frac{1}{2} (h_{i+\frac{1}{2}})^{-2} \right]. \quad (4.6)$$

Since

$$\vartheta_3([i+\tfrac{1}{2}]\Delta t) = \vartheta_3(i\Delta t) + \tfrac{1}{2} (\Delta t)(h_{i+\frac{1}{2}})^{-3} + O([\Delta t]^2),$$

$$\vartheta_2([i+\tfrac{1}{2}]\Delta t) = \vartheta_2(i\Delta t) + \tfrac{1}{2} (\Delta t)(h_{i+\frac{1}{2}})^{-2} + O([\Delta t]^2),$$

it follows from (2.4), (4.5), and (4.6) that

$$|(\Delta t)\Psi_i - \psi([i+\tfrac{1}{2}]\Delta t)| \le K_3(\Delta t)^2. \quad (4.7)$$

Remembering that $\tau > 1$ (see Theorem 2.4 and Lemma 2.1) we see from (1.6) and (2.6) that $\psi'(\tau) > 0$. Hence, since $\psi(\tau) = 0$,

$$\left.\begin{aligned} \psi(t) &\ge K_4|t-\tau|, & t &\in [\tau, \tau + (\Delta t)_1], \\ \psi(t) &\le -K_4|t-\tau|, & t &\in [\tau - (\Delta t)_1, \tau]. \end{aligned}\right\} \quad (4.8)$$

Let

$$(\Delta t)_2 = \min \{(\Delta t)_1/2, (T-\tau)/4, K_4/(16K_3)\}, \quad (4.9)$$

and denote by $n_1 = n_1(\Delta t)$ the integer such that

$$(n_1 + \tfrac{1}{2})\Delta t \le \tau - (\Delta t)/8 < (n_1 + 3/2)\Delta t. \quad (4.10)$$

From equations (4.7) through (4.10) we have that

$$(\Delta t)\,\Psi_{n_1} \le \left|(\Delta t)\,\Psi_{n_1} - \psi([n_1+\tfrac{1}{2}]\Delta t)\right| + \psi([n_1+\tfrac{1}{2}]\Delta t),$$

$$\le K_3(\Delta t)^2 - K_4\left|(n_1+\tfrac{1}{2})\Delta t - \tau\right| ,$$

$$\le K_3(\Delta t)^2 - K_4(\Delta t)/8,$$

$$= K_3(\Delta t)\,[\Delta t - K_4/(8K_3)] ,$$

$$< 0 , \text{ for } \Delta t \le (\Delta t)_2 . \qquad (4.11)$$

It follows from (4.10) that

$$\tau + 2\Delta t > (n_1 + 5/2)\,\Delta t > \tau + (\Delta t)/8 .$$

Hence, from equations (4.7) through (4.9) we have that

$$(\Delta t)\Psi_{n_1+2} \ge -\left|(\Delta t)\Psi_{n_1+2} - \psi([n_1+5/2]\Delta t)\right| + \psi([n_1+5/2]\Delta t),$$

$$\ge -K_3(\Delta t)^2 + K_4\left|(n_1+5/2)\Delta t - \tau\right|,$$

$$\ge -K_3(\Delta t)^2 + K_4(\Delta t)/8 ,$$

$$= K_3(\Delta t)\,[-(\Delta t) + K_4/(8K_3)] ,$$

$$> 0, \text{ for } \Delta t \le (\Delta t)_2 . \qquad (4.12)$$

Finally, if

$$\tau + (\Delta t)/8 \le (n_1 + 3/2)\,\Delta t , \qquad (4.13)$$

then,

$$(\Delta t)\Psi_{n_1+1} \geq - \left| (\Delta t)\Psi_{n_1+1} - \psi([n_1 + 3/2]\Delta t) \right| + \psi([n_1 + \tfrac{3}{2}\Delta t]),$$

$$\geq -K_3(\Delta t)^2 + K_4 \left| (n_1 + 3/2)\Delta t - \tau \right| ,$$

$$\geq -K_3(\Delta t)^2 + K_4(\Delta t)/8 ,$$

$$= K_3(\Delta t)\ [-\Delta t + K_4/(8K_3)] ,$$

$$> 0, \quad \text{for} \quad \Delta t \leq (\Delta t)_2 . \tag{4.14}$$

If n is defined as in Lemma 3.1, it follows from Lemma 3.1, Theorem 3.4, (4.11), and (4.12), that

$$m = n + 1 = \begin{cases} n_1 + 1, & \text{if } \Psi_{n_1+1} > 0 , \\ n_1 + 2, & \text{if } \Psi_{n_1+1} \leq 0 , \end{cases} \tag{4.15}$$

for $\Delta t \leq (\Delta t)_2$.

We now prove that

$$|\tau - m\Delta t| \leq 5(\Delta t)/8, \quad \text{for} \quad \Delta t \leq (\Delta t)_2 . \tag{4.16}$$

Case 1: $m = n_1 + 1$

Then, using (4.10),

$$|m\Delta t - \tau| = |(n_1 + 1)\Delta t - \tau| ,$$

$$\leq |(n_1 + 1)\Delta t - [\tau - (\Delta t)/8]| + (\Delta t)/8 ,$$

$$\leq (\Delta t)/2 + (\Delta t)/8 ,$$

$$= 5(\Delta t)/8 .$$

Case 2: $m = n_1 + 2$

Then $\Psi_{n_1+1} \leq 0$. Since (4.13) implies (4.14), it follows that (4.13) does not hold. Hence, noting (4.10) ,

$$(n_1 + 3/2)\Delta t < \tau + (\Delta t)/8 < (n_1 + 5/2)\Delta t .$$

Therefore,

$$|m\Delta t - \tau| = |(n_1 + 2)\Delta t - \tau| ,$$

$$\leq |(n_1 + 2)\Delta t - [\tau + (\Delta t)/8]| + (\Delta t)/8 ,$$

$$\leq (\Delta t)/2 + (\Delta t)/8 ,$$

$$= 5(\Delta t)/8 ,$$

We have thus established (4.16).

We now consider (4.4). First, we note that $\varphi'(\tau) = 0$ (see Theorem 2.4, Lemma 2.1, and (2.5)). Thus,

$$|\varphi(t) - \varphi(\tau)| \leq K_5 |t - \tau|^2 . \qquad (4.17)$$

Next, we see from (1.5), (2.2), (4.5), and (4.16), that

$$|(\Delta t)I_3(m)|, \ |\vartheta_3(m\Delta t)| \geq K_6 > 0, \quad \text{for} \quad \Delta t \leq (\Delta t)_3 . \qquad (4.18)$$

Hence, using (2.3), (3.7), and (4.5),

$$\begin{aligned} |\Phi_m - \varphi(m\Delta t)| = \\ & |\{[(\Delta t)I_2(m) - \vartheta_2(m\Delta t)]\ \vartheta_3(m\Delta t) + \\ & + [\vartheta_3(m\Delta t) - (\Delta t)\, I_3(m)]\ \vartheta_2(m\Delta t)\}/\{(\Delta t)I_3(m)\vartheta_3(m\Delta t)\}| \\ & \leq K_7 (\Delta t)^2, \quad \text{for} \quad \Delta t \leq (\Delta t)_3 . \end{aligned} \qquad (4.19)$$

It follows from (4.16), (4.17), and (4.19), that

$$|\Phi_m - \varphi(\tau)| \leq K_8 (\Delta t)^2, \quad \text{for} \quad \Delta t \leq (\Delta t)_4 . \qquad (4.20)$$

Since $p(\tau) = p'(\tau) = 0$ (see (1.3) and (1.4)),

$$|p(t)| \leq K_9 |t - \tau|^2 . \qquad (4.21)$$

Finally, it follows from (1.5), (3.16), (3.20), (4.16), and (4.20), that

$$
\begin{aligned}
|\nabla P_m| &= |\nabla Q_m(m)|, \\
&= \left|(\Delta t)[h_{m-\frac{1}{2}} - \Phi_m](h_{m-\frac{1}{2}})^{-3}\right| , \\
&= \left|(\Delta t)\{[h_{m-\frac{1}{2}} - h(\tau)] + [h(\tau) - \varphi(\tau)] \right. \\
&\qquad \left. + [\varphi(\tau) - \Phi_m]\}(h_{m-\frac{1}{2}})^{-3}\right|, \\
&\le (\Delta t)\{K_{11}|\tau - (m-\tfrac{1}{2})\Delta t| + K_8(\Delta t)^2\}(h_{m-\frac{1}{2}})^{-3} , \\
&\le K_{12}(\Delta t)^2, \quad \Delta t \le (\Delta t)_4 . \qquad (4.22)
\end{aligned}
$$

Now let $n_2 = n_2(\Delta t)$ be the largest integer such that

$$n_2 \Delta t \le \min \{m\Delta t, \tau\} . \qquad (4.23)$$

Then it follows from (4.16) that

$$\max\{m\Delta t, \tau\} < (n_2 + 2)\Delta t; \quad \Delta t \le (\Delta t)_2 . \qquad (4.24)$$

We now assert that (4.3) and (4.4) hold if we set

$$(\Delta t)_0 = \min\{(\Delta t)_2, (\Delta t)_3, (\Delta t)_4\} .$$

Clearly, (4.16) implies (4.3). To prove (4.4) it is necessary to consider three cases.

Case 1: $n_2 + 2 \leq j \leq N$

Then

$$p(j\Delta t) = P_j = 0 . \tag{4.25}$$

Case 2: $0 \leq j \leq n_2$

Then, using (2.14) and (3.20),

$$\begin{aligned} |p(j\Delta t) - P_j| &= |q(j\Delta t;\tau) - Q_j(m)| , \\ &= |\vartheta_2(j\Delta t) - (\Delta t)\, I_2(j) - \varphi(\tau)\vartheta_3(j\Delta t) + (\Delta t)\, \Phi_m\, I_3(j)| , \\ &\leq |\vartheta_2(j\Delta t) - (\Delta t)\, I_2(j)| + |[\Phi_m - \varphi(\tau)]\, (\Delta t)\, I_3(j)| \\ &\quad + |\varphi(\tau)\, [(\Delta t)\, I_3(j) - \vartheta_3(j\Delta t)| , \\ &\leq \{K_1 + K_8[(\Delta t)\, I_3(m)] + |\varphi(\tau)|K_2\}(\Delta t)^2 . \end{aligned} \tag{4.26}$$

Case 3: $j = n_2 + 1$

Then, using (4.21), (4.23), and (4.24),

$$|p(j\Delta t)| \leq \begin{cases} K_9(\Delta t)^2, & \text{if } j\Delta t \leq \tau , \\ 0 , & \text{if } j\Delta t > \tau . \end{cases} \tag{4.27}$$

On the other hand, using (4.22), (4.23), and (4.24),

$$|P_j| \le \begin{cases} K_{12}(\Delta t)^2, & \text{if } j < m, \\ 0, & \text{if } j \ge m. \end{cases} \tag{4.28}$$

Clearly, equations (4.25) through (4.28) imply (4.4) for some K . The proof of Theorem 4.1 is therefore complete.

We conclude this section with two observations. Firstly, we draw the reader's attention to the fact that $\{\underline{P}, m\}$ is as accurate an approximation to $\{p(t), \tau\}$ as could be hoped for. For we can at best have that

$$|\tau - m\Delta t| \le (\Delta t)/2 , \tag{4.29}$$

and (4.3) is almost as good as (4.29). Since we can at best have (4.29), we might expect that $|p(j\Delta t) - P_j| = O(\Delta t)$. Instead, the gods have smiled and we have (4.4).

Secondly, as the reader may have noticed, if we replaced (4.10) by the condition

$$(n_1 + \tfrac{1}{2})\Delta t \le \tau - (\Delta t)/2r < (n_1 + 3/2)\Delta t ,$$

for any integer r , $r \ge 4$, then we can use the same method of proof as in Theorem 4.1 to prove that

$$|\tau - m\Delta t| \le (\Delta t)(r+1)/2r .$$

5. Another Discrete Approximation

In this section we formulate a second discrete approximation to Problem 2, Problem 3D, and prove that Problems 2D and 3D are equivalent. The reason for introducing Problem 3D is that the iterative procedure in step 2 of Christopherson's method is best understood if it is regarded as an algorithm for solving Problem 3D.

We set $M = N - 1$ and denote by $\underline{A}$ the $M \times M$ matrix with components

$$A_{ij} = \begin{cases} -(h_{i+\frac{1}{2}})^3, & \text{if } j = i+1, \\ [(h_{i+\frac{1}{2}})^3 + (h_{i-\frac{1}{2}})^3], & \text{if } j = i, \\ -(h_{i-\frac{1}{2}})^3, & \text{if } j = i-1, \\ 0, & \text{otherwise}, \end{cases} \tag{5.1}$$

for $1 \le i,\ j \le M$. Noting (1.5), we see that $\underline{A}$ is a symmetric irreducibly diagonally dominant matrix with positive diagonal entries, so that $\underline{A}$ is positive definite (Varga [11, p. 23]).

We denote by $\underline{B}$ the M-vector with components

$$B_i = -(\Delta t)\,[h_{i+\frac{1}{2}} - h_{i-\frac{1}{2}}], \quad 1 \le i \le M. \tag{5.2}$$

We can now formulate

Problem 3D

Find M-vectors $\underline{X}$ and $\underline{Y}$ such that

$$\underline{A}\,\underline{X} - \underline{Y} = \underline{B}\,, \tag{5.3}$$

$$\underline{X}^T\underline{Y} = 0\,, \tag{5.4}$$

$$\underline{X} \geq 0, \quad \underline{Y} \geq 0\,. \tag{5.5}$$

Theorem 5.1

Problem 3D has a unique solution.

Problems 2D and 3D are equivalent. If $\{\underline{P}, m\}$ and $\{\underline{X}, \underline{Y}\}$ are the solutions of Problems 2D and 3D, respectively, then

$$X_i = P_i\,, \quad 1 \leq i \leq M\,, \tag{5.6}$$

$$\underline{Y} = \underline{A}\,\underline{X} - \underline{B}\,, \tag{5.7}$$

$$P_0 = P_N = 0\,, \tag{5.8}$$

$$m = \inf\{i; Y_i > 0\} \tag{5.9}$$

$$Y_i = 0\,, \quad 1 \leq i < m\,, \tag{5.10}$$

$$Y_i > 0\,, \quad m \leq i \leq M\,, \tag{5.11}$$

$$X_i > 0\,, \quad 1 \leq i \leq m-2\,, \tag{5.12}$$

$$X_{m-1} = -(\Delta t)\Psi_{m-1} / \{(h_{m-\frac{1}{2}})^3 I_3(m)\} \geq 0\,, \tag{5.13}$$

$$X_i = 0\,, \quad m \leq i \leq M\,. \tag{5.14}$$

Proof: Since $\underline{A}$ is positive definite it is known from the theory of quadratic programming (see Cryer [4]) that Problem 3D has a unique solution.

Let $\{\underline{P}, m\}$ be the solution of Problem 2D, and let $\underline{X}$ and $\underline{Y}$ be defined by (5.6) and (5.7). We assert that (5.9) through (5.14) hold.

First we establish (5.12) through (5.14). Equation (5.14) follows immediately from (3.16), (3.18), and (5.6). To prove (5.13), we note that

$$
\begin{aligned}
P_{m-1} &= Q_{m-1}(m) ,\\
&= \Delta t \{I_2(m-1) - \Phi_m I_3(m-1)\} ,\\
&= -(\Delta t)(\Delta \Phi_{m-1}) I_3(m-1) ,\\
&= -(\Delta t) \Psi_{m-1}/\{(h_{m-\frac{1}{2}})^3 I_3(m)\} .
\end{aligned}
$$

Finally, (5.12) follows from (3.21), (3.19), and Lemma 3.2.

Next, we establish (5.9) through (5.11). First it follows from (3.1), (5.1), (5.2), and (5.7) that

$$Y_i = -(L\underline{P})_i(\Delta t)^2, \qquad 1 \le i \le m-1 .$$

Noting (3.13) and (3.20), we see that (5.10) holds. To prove (5.11) we observe that it follows from (5.7) and (5.14) that

$$Y_i = \begin{cases} A_{m,m-1} X_{m-1} - B_m, & i = m , \\ -B_i , & m+1 \le i \le M . \end{cases} \tag{5.15}$$

But from Lemma 3.1 and (3.19) we see that $(m + \frac{1}{2})\Delta t \ge 1$. Noting (1.6), if follows that

$$B_i < 0 , \quad m+1 \le i \le M . \tag{5.16}$$

On the other hand, using (5.13), we have

$$\begin{aligned} & A_{m,m-1} X_{m-1} - B_m \\ & \qquad = (\Delta t) \ \{\Psi_{m-1}/I_3(m) + [h_{m+\frac{1}{2}} - h_{m-\frac{1}{2}}]\} , \\ & \qquad = (\Delta t) \ \{[h_{m-\frac{1}{2}} \ I_3(m-1) - I_2(m-1)] \\ & \qquad\qquad + I_3(m) \ [h_{m+\frac{1}{2}} - h_{m-\frac{1}{2}}]\} \ / \ I_3(m) , \\ & \qquad = (\Delta t) \ \{h_{m+\frac{1}{2}} \ I_3(m) - I_2(m-1) \\ & \qquad\qquad - h_{m-\frac{1}{2}} [I_3(m) - I_3(m-1)]\}/I_3(m), \\ & \qquad = (\Delta t) \ \Psi_m/I_3(m) . \end{aligned} \tag{5.17}$$

Since $\Psi_m > 0$ (see Lemma 3.1 and (3.19)), equation (5.11) follows from (5.15), (5.16), and (5.17). Finally, (5.9) follows from (5.10) and (5.11) .

We can now show that Problems 2D and 3D are equivalent. We have seen that if $\{\underline{P}, m\}$ is the solution of Problem 2D and $\underline{X}$ and $\underline{Y}$ are defined by (5.6) and (5.7) then (5.9) through (5.14) hold. But (5.7) and (5.10) through (5.14) imply (5.3) through (5.5), so that $\{\underline{X}, \underline{Y}\}$ is a solution of Problem 3D. Remembering that the solutions of Problems 2D and 3D are unique, and noting that equations (5.6) through (5.9) define a one-to-one correspondence between the pairs $\{\underline{P}, m\}$ and $\{\underline{X}, \underline{Y}\}$, the equivalence of Problems 2D and 3D follows.

6. The Iterative Solution of the Discrete Approximation

In this section we analyse the algorithm used by Christopherson to compute the solution of the discrete approximations to Problem 1 .

Algorithm 6.1

Choose an M-vector $\underline{X}^0 = \{X_i^{(0)}\}$ where $\underline{X}^{(0)} \geq 0$. Choose a relaxation parameter ω, where $0 < \omega < 2$.

Generate a sequence of M-vectors $\underline{X}^{(k)} = \{X_i^{(k)}\}$, $\underline{R}^{(k)} = \{R_i^{(k)}\}$, and $\underline{Y}^{(k)} = \{Y_i^{(k)}\}$, $k = 1, 2, \ldots,$ using the equations,

$$R_i^{(k+1)} = B_i - \sum_{j=1}^{i-1} A_{ij} X_j^{(k+1)} - \sum_{j=1}^{M} A_{ij} X_j^{(k)} , \qquad (6.1)$$

$$X_i^{(k+1)} = \max \{ 0,\ X_i^{(k)} + \omega R_i^{(k+1)}/A_{ii} \} , \qquad (6.2)$$

$$Y_i^{(k+1)} = -R_i^{(k+1)} + A_{ii} (X_i^{(k+1)} - X_i^{(k)}) . \qquad (6.3)$$

The reader will have observed that Algorithm 6.1 consists of applying S.O.R. (systematic overrelaxation) to the equations $\underline{A}\,\underline{X} = \underline{B}$ with the proviso that the iterates $\underline{X}^{(k)}$ be non-negative. This was the way in which Algorithm 6.1 was viewed by Christopherson except that, since he worked by hand, he used relaxation rather than S.O.R. The condition that the vectors $\underline{X}^{(k)}$ be non-negative arises naturally from the physical restraint that the lubricating fluid cannot support negative pressures.

Christopherson used Algorithm 6.1 without explicitly formulating the discrete problem that he was solving. Of the two formulations of the discrete problem that we have developed, Problems 2D and 3D, it seems to us that Problem 3D lies closest in spirit to Christopherson's ideas.

In computations for Cameron and Wood [2], Fox (working by hand) used Algorithm 6.1 with relaxation instead of S.O.R.; Raimondi and Boyd [10] (using an IBM 704) used the Liebmann or Gauss-Seidel method instead of S.O.R.; finally, the use of S.O.R. was suggested by Gnanadoss and Osborne [5].

Throughout the remainder of this section we denote the solutions of Problems 2D and 3D by $\{\underline{P}, m\}$ and $\{\underline{X}, \underline{Y}\}$, respectively, and assume that $\underline{X}^{(k)}$ and $\underline{Y}^{(k)}$ are generated using Algorithm 1.

First we show that Algorithm 6.1 is always convergent:

<u>Theorem 6.1</u>

For any $\underline{X}^{(0)} \geq 0$, $\underline{X}^{(k)} \to \underline{X}$ and $\underline{Y}^{(k)} \to \underline{Y}$ as $k \to \infty$.

<u>Proof:</u> Since $\underline{A}$ is positive definite, the theorem follows from Theorem 3.1 of Cryer [4].

Next we consider the speed with which Algorithm 6.1 converges. We define the <u>asymptotic rate of convergence</u> of Algorithm 6.1 to be

$$R(\underline{A}, \underline{B}, \omega) = -\log \{ \sup_{\underline{X}^{(0)} \geq 0} \limsup_{k \to \infty} \|\underline{X}^{(k)} - \underline{X}\|^{1/k} \}, \qquad (6.4)$$

where $\|\cdot\|$ denotes any vector norm.

We need certain concepts from the theory of S.O.R. (see Varga [11]). Let $\underset{\sim}{\tilde{A}}$ be a $p \times p$ positive definite matrix. Let $\tilde{\underline{A}} = \tilde{\underline{D}} - \tilde{\underline{E}} - \tilde{\underline{F}}$, where $\tilde{\underline{D}}$ is a diagonal matrix while $\tilde{\underline{E}}$ and $\tilde{\underline{F}}$ are respectively strictly upper and strictly lower triangular matrices. Then the <u>point successive relaxation matrix</u> corresponding to $\tilde{\underline{A}}$ is given by

$$\mathcal{L}_\omega(\tilde{\underline{A}}) = (\tilde{\underline{D}} - \omega\tilde{\underline{E}})^{-1} \{(1-\omega)\tilde{\underline{D}} + \omega\tilde{\underline{F}}\} . \tag{6.5}$$

The <u>point Jacobi matrix</u> $\tilde{\underline{J}}$ is given by

$$\tilde{\underline{J}} = (\tilde{\underline{D}})^{-1} [\tilde{\underline{E}} + \tilde{\underline{F}}] . \tag{6.6}$$

The <u>asymptotic rate of convergence</u> for $\mathcal{L}_\omega(\tilde{\underline{A}})$ is given by

$$R_\infty[\mathcal{L}_\omega(\tilde{\underline{A}})] = -\log \{\rho[\mathcal{L}_\omega(\tilde{\underline{A}})]\}, \tag{6.7}$$

where $\rho[\mathcal{L}_\omega(\tilde{\underline{A}})]$ is the spectral radius of $\mathcal{L}_\omega(\tilde{\underline{A}})$. Finally, the <u>optimum relaxation parameter</u> $\omega_b = \omega_b(\tilde{\underline{A}})$ satisfies

$$R_\infty[\mathcal{L}_{\omega_b}(\tilde{\underline{A}})] = \max_\omega R_\infty[\mathcal{L}_\omega(\tilde{\underline{A}})] . \tag{6.8}$$

If $\tilde{\underline{A}}$ is consistently ordered and two-cyclic then

$$\omega_b(\tilde{\underline{A}}) = 2/\{1 + [1 - \rho^2(\tilde{\underline{J}})]^{\frac{1}{2}}\} . \tag{6.9}$$

Theorem 6.2

Assume that

$$\Psi_{m-1} > 0 . \tag{6.10}$$

Then

$$R(\underline{A}, \underline{B}, \omega) = R_{\infty}(\mathcal{L}_{\omega}[\underline{A}^{(m-1)}]), \tag{6.11}$$

where $\underline{A}^{(m-1)}$ is the $(m-1) \times (m-1)$ matrix consisting of the first $(m-1)$ rows and columns of $\underline{A}$. Moreover, if

$$\omega_{opt} = \omega_b(\underline{A}^{(m-1)}) \tag{6.12}$$

then

$$R(\underline{A}, \underline{B}, \omega) \leq R(\underline{A}, \underline{B}, \omega_{opt}), \quad 0 < \omega < 2 , \tag{6.13}$$

and

$$\omega_{opt} \leq \omega_b(\underline{A}) . \tag{6.14}$$

Proof: From (5.10) through (5.13), we have

$$X_i + Y_i > 0 , \quad 1 \leq i \leq M .$$

Equation (6.11) now follows by Theorem 4.2 of Cryer [4].

Inequality (6.13) follows from (6.11), (6.12), and (6.8) .

Finally, to prove (6.14) let $\underline{J}$ and $\underline{J}^{(m-1)}$ be the point Jacobi matrices corresponding to $\underline{A}$ and $\underline{A}^{(m-1)}$ respectively. Let $\hat{\underline{J}}$ be the $M \times M$ matrix obtained by augmenting $\underline{J}^{(m-1)}$ by $(M - m + 1)$ rows and columns of zeros. Then

$$\rho(\underline{J}^{(m-1)}) = \rho(\hat{\underline{J}}) . \tag{6.15}$$

Also, by the Perron-Frobenius theory (Varga [11, p. 30]),

$$\rho(\hat{\underline{J}}) \leq \rho(\underline{J}) . \tag{6.16}$$

Since $\underline{A}$ and $\underline{A}^{(m-1)}$ are consistently ordered two-cyclic matrices, (6.14) follows from (6.15), (6.16), and (6.9).

7. Numerical Results

In this section we present numerical results for an infinitely long full journal bearing to illustrate the theoretical results of the preceding sections.

The equations for an infinitely long full journal bearing are (Pinkus and Sternlicht [9, p. 42 and p. 46])

$$\frac{d}{d\theta}\left[w^3(\theta)\frac{dp}{d\theta}\right] = \frac{dw(\theta)}{d\theta}, \quad 0 < \theta < \theta_2, \tag{7.1}$$

$$p(\theta) = 0, \quad \theta_2 \le \theta \le 2\pi, \tag{7.2}$$

$$p(0) = 0, \tag{7.3}$$

$$\frac{d}{d\theta}p(\theta) = 0, \quad \theta = \theta_2, \tag{7.4}$$

$$w(\theta) = 1 + \epsilon\cos\theta. \tag{7.5}$$

Here, ϵ is the eccentricity ratio and satisfies $0 \le \epsilon < 1$.

Introducing the variable

$$t = \theta/\pi, \tag{7.6}$$

it is found that p satisfies equations (1.1) through (1.4) with $T = 2$, $\tau = \theta_2/\pi$, and

$$h(t) = (1 + \epsilon\cos\pi t)/\sqrt{\pi}. \tag{7.7}$$

It is easily verified that h satisfies (1.5), (1.6), (1.7), (3.4), and (4.1), so that all the results of the preceding sections are valid.

For h given by (7.7) the solution of Problem 1 can be given analytically. Using the results in Pinkus and Sternlicht [9, p. 47] we find that

$$\tau = (\alpha + \pi)/\pi \,, \tag{7.8}$$

$$p(t) = \frac{1}{[1-\epsilon^2]^{3/2}} \left\{ \gamma(t) - \epsilon \sin \gamma(t) - \frac{[(2+\epsilon^2)\gamma(t) - 4\epsilon \sin \gamma(t) + \epsilon^2 \sin \gamma(t) \cos \gamma(t)]}{2(1+\epsilon \cos \beta)} \right\} \tag{7.9}$$

for $0 < t < \tau$,

where β is the zero in $[0, \pi/2]$ of

$$F(x) = \epsilon[\sin x \cos x - (\pi + x)] + 2[(\pi + x)\cos x - \sin x], \tag{7.10}$$

$$\alpha = \text{arc cos}\, \{(\epsilon + \cos \beta)/(1 + \epsilon \cos \beta)\}\,, \tag{7.11}$$

and

$$\gamma(t) = \begin{cases} \text{arc cos}\, \{[\epsilon + \cos(\pi t)]/[1 + \epsilon \cos(\pi t)]\}, & 0 \le t \le 1\,, \\ 2\pi - \text{arc cos}\, \{[\epsilon + \cos(\pi t)]/[1 + \epsilon \cos(\pi t)]\}\,, & 1 \le t \le 2. \end{cases} \tag{7.12}$$

As shown in Theorem 6.2, if $\Psi_{m-1} > 0$ then $\omega_{opt} \leq \omega_b(\underline{A})$. Let $\underline{J}$ be the point Jacobi matrix corresponding to $\underline{A}$. Then, from (6.9),

$$\omega_b(\underline{A}) = 2/\{1 + [1 - [\rho(\underline{J})]^2]^{\frac{1}{2}}\}. \tag{7.13}$$

Now $\underline{J}$ is similar to the $M \times M$ tridiagonal symmetric matrix $\underline{\tilde{J}}$ with components

$$\tilde{J}_{ij} = \begin{cases} f_{i+\frac{1}{2}}, & \text{if } j = i+1, \\ f_{i-\frac{1}{2}}, & \text{if } j = i-1, \\ 0, & \text{otherwise}, \end{cases}$$

where

$$f_{i+\frac{1}{2}} = (h_{i+\frac{1}{2}})^3 \{(h_{i+\frac{1}{2}})^3 + (h_{i+\frac{3}{2}})^3\}^{-\frac{1}{2}} \{(h_{i-\frac{1}{2}})^3 + (h_{i+\frac{1}{2}})^3\}^{-\frac{1}{2}}.$$

Hence, for small Δt, $\underline{\tilde{J}}$ is approximately equal to $\underline{\hat{J}}$ where

$$\hat{J}_{ij} = \begin{cases} \frac{1}{2}, & \text{if } |i-j| = 1, \\ 0, & \text{otherwise}. \end{cases}$$

It is well known that the largest eigenvalue of $\underline{\hat{J}}$ is equal to $\cos(\pi/N)$ and that the corresponding eigenvector has components $\sin(\pi i/N)$. Replacing $\rho(\underline{J})$ by $\rho(\underline{\hat{J}})$ in (7.13), we obtain

$$\omega_b(\underline{A}) \approx \omega^* = 2/\{1 + \sin(\pi/N)\} . \qquad (7.14)$$

We do not claim that ω^* is a good approximation to $\omega_b(\underline{A})$ since the difference between $\rho(\underline{J})$ and $\rho(\hat{\underline{J}})$ is "magnified" by (7.13). Nevertheless, noting (6.12), (6.13), and (6.14), and remembering that it is in general better to use a value of ω which is too large rather than too small (Varga [11, p. 114]) we believe that is is a reasonable strategy to set $\omega = \omega^*$.

Numerical results were obtained for the case $\epsilon = .8$. The computations were performed on the UNIVAC 1108 computer at the University of Wisconsin; this computer uses eight-decimal floating point arithmetic. The analytic solution $\{p(t), \tau\}$ was computed using equations (7.8) through (7.12), the zero, β , of $F(x)$ being computed to eight decimals by the method of interval-halving. The discrete approximation $\{\underline{P}, m\}$ was computed by solving Problem 3D using Algorithm 6.1; the iterations were terminated when

$$\|\underline{R}^{(k+1)}\|_\infty = \max_i |R_i^{(k+1)}| \le 10^{-7} . \qquad (7.15)$$

The initial approximation $\underline{X}^{(0)}$ was always taken to be identically zero.

Two experiments were carried out. In the first experiment, N was taken equal to 64 while ω was varied. We were primarily interested in determining the number of iterations required to converge, that is the number of iterations required before (7.15) was satisfied. The results (see Table 7.1) reinforced our opinion that while setting $\omega \mp \omega^*$ does not ensure optimum convergence, it is a reasonable strategy to adopt.

ω	No. of iterations to converge
1.0	811
1.1	670
1.2	558
1.3	453
1.4	362
1.5	288
1.6	216
1.7	146
1.8	70
1.9	136
$\omega^* = 1.90645$	146

Table 7.1

Number of iterations to converge ($N = 64$).

In the second experiment, ω was taken equal to ω^* while N was varied. We were primarily interested in the difference between $\{p, \tau\}$ and $\{\underline{P}, m\}$. Setting

$$\|p - \underline{P}\|_\infty = \max_j |p(j\Delta t) - P_j| , \tag{7.16}$$

the dependence of $\|p - \underline{P}\|_\infty$ upon Δt is shown in Table 7.2.

N	Δt	$\|p - \underline{P}\|_\infty$	No. of iterations to converge	No. of iterations to stabilize sign pattern
64	.03125	.016017	146	9
128	.015625	.002725	268	14
256	.0078125	.000756	513	56
512	.00390625	.000170	923	319
1024	.001953125	.000073	1640	714

Table 7.2

Dependence of $\|p - \underline{P}\|_\infty$ upon Δt .

Bearing in mind that the UNIVAC 1108 works to only eight decimal places, it is clear that the results are in agreement with the assertion of Theorem 4.1 that $\|p - \underline{P}\|_\infty \le K(\Delta t)^2$. The other assertion of Theorem 4.1, namely that $|\tau - m\Delta t| \le 5(\Delta t)/8$, was always satisfied.

Finally, in the second experiment we also observed how the "sign pattern" of $\underline{Y}^{(k)}$ varied with k . Let

$$m^{(k)} = \inf \{ i ; Y_i^{(k)} \} > 0 .$$

We found that $m^{(k)}$ was a good approximation to m for quite small k . For example, for N = 1024, m = 569, $m^{(50)} = 567$, and $m^{(k)} = 569$ for $k \ge 714$. It follows from this observation that it might be possible to reduce the number of iterations required to converge by first carrying out a small number of iterations with $\omega = \omega^*$ so as to obtain a good estimate for m , and then using (6.12) (rather than (6.14)) to estimate ω_{opt} .

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13. ABSTRACT

A method for solving free boundary problems for journal bearings by means of finite differences has been proposed by Christopherson. We analyse Christopherson's method in detail for the case of an infinite journal bearing where the free boundary problem is as follows: Given $T > 0$ and $h(t)$ find $\tau \in (0, T]$ and $p(t)$ such that (i) $[h^3p']' = h'$ for $t \in (0,\tau)$, (ii) $p(0) = 0$, (iii) $p(t) = 0$ for $t \in [\tau, T]$, and (iv) $p'(\tau-0) = 0$.

First it is shown that the discrete approximation is accurate to $O([\Delta t]^2)$ where Δt is the stepsize. Next it is shown that the discrete problem is equivalent to a quadratic programming problem. Then the iterative method for computing the discrete approximation is analysed. Finally, some numerical results are given.

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