Use of 4-D Diagrams to Solve Equations

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1. Introduction

When any polynomial is over the complex numbers it is known that it will have a number of solutions equal to the degree of the polynomial. The situation is very different when a degree n polynomial is over $k \times k$ matrices. In this case there are either 0, infinite, or somewhere between 1 and $\binom{kn}{k}$ solutions. We are interested in determining which numbers of solutions are possible for degree n polynomials over $k \times k$ matrices.

It is easy to construct polynomials of any degree and with matrices of any size that have either zero or infinitely many solutions. It is also possible to construct a polynomial with a number of solutions not equal to the degree.

Number of solutions is not equal to the degree:

Consider $X^2 - \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} = 0$. How many solutions exist?

Solution: By combining the matrices using matrix operations, one can form the equation

$$\begin{bmatrix} a^2+bc-4 & ab+bd \\ ca+dc & bc+d^2-9 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

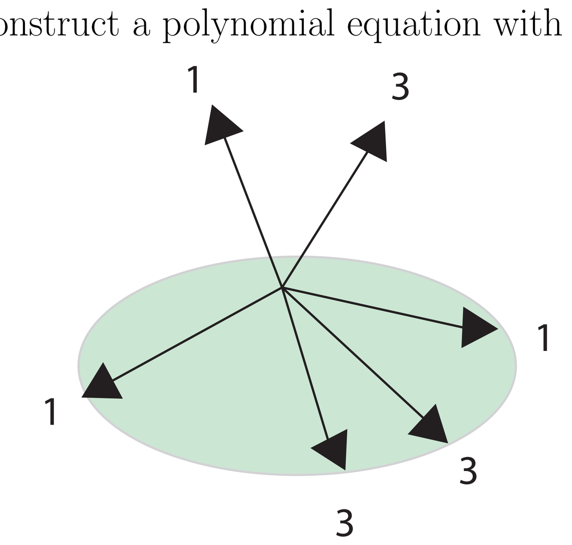
Thus, there are four variables and four equations, so one can solve this system of equations to arrive at the solution set: $X = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$. Hence, with a second degree polynomial, it is possible to have 4 unique solutions.

2. Diagram Representations

When given a degree and size for the matrices, it is generally difficult to construct a polynomial with a given number of solutions. To simplify this, we use a diagram of vectors and using this diagram construct a polynomial with diagonalizable solutions. We are only considering allowable diagrams. An *allowable diagram* is one with kn values, each vector can have at most n critical values, and each vector has a distinct eigenvalue.

Thm [1]: For an allowable diagram, there exists an equation with diagonalizable solutions that corresponds to the diagram.

We have been focusing on degree 3 polynimials, $n = 3$, with 4×4 matrices, $k = 4$. This requires us to use 4 dimensional diagrams to construct our equations. For example, this allowable diagram will construct a polynomial equation with 189 diagonalizable solutions.



3. Our Questions

This previous background has brought us to investigate three main questions.

1. How many solutions are possible to achieve with diagonalizable matrices?

It has been shown that for $k = 2$ it is possible to construct a polynomial equation with anywhere between 0 and $\binom{kn}{k}$ solutions. We have found that in the case $n = 2, k = 3$ 11 is the only number between 0 and 20 that is impossible to obtain. We have also found all possible numbers of solutions for $k = 4$ and $n = 2$. We believe that with $n \geq k$ all possible numbers of solutions can be obtained with diagonalizable matrices.

2. Are the solutions not attainable with diagonalizable attainable with nondiagonalizable matrices?

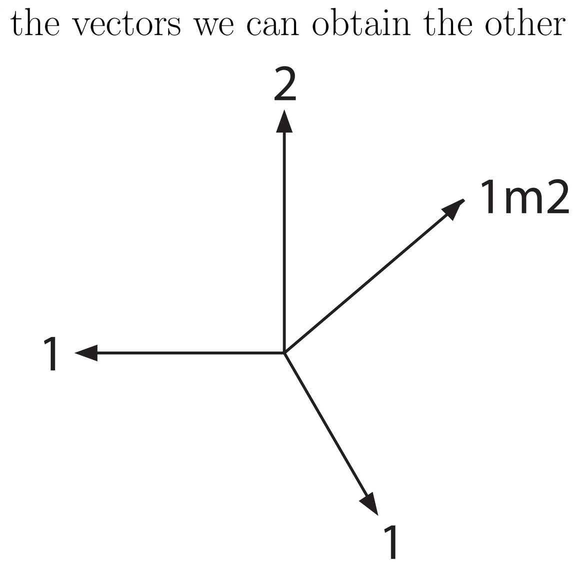
Since we know there are some numbers of solutions that cannot be attained using only diagonalizable matrices, can we always construct a polynomial equation with those numbers of solutions using non-diagonalizable matrices? It is possible in the case for 11, and we believe that it is possible in general.

3. For a given diagram, the numbers of non-diagonalizable solutions are specific values within a range. Which values in the range can be obtained?

For example, the diagram used to construct 7 diagonalizable solutions has 9, 10, and 11 as possible numbers of solutions when using non-diagonalizable matrices.

4. Question 3

By examining the diagram we can determine the minimum and maximum numbers of solutions. The miniumum number of solutions is always the number attainable with only diagonalizable matrices. The maximum number is constructed using non-diagonalizable matrices. By wisely selecting our variables, we can reduce the number of solutions. Using the maximum number of solutions and the values of the vectors we can obtain the other numbers of solutions possible.



This diagram gives a minimum of 7 solutions and a maximum of 11 solutions. It is also possible to get 9 and 10 solutions, but not 8.

5. Question 1 and Finite Fields

In attempting to answer this question, we have usually focused on trying to find all possible solution numbers for a given k and n. This is difficult because when using real numbers as entries in the matrices there are an infinite number of possible equations. To combat this problem we have been using matrices over finite fields, which gives us a finite number of possible equations. Since there are a finite number of possible equations, we can compute the number of solutions for every equation and thus determine which numbers of solutions are possible. A *finite field* is a finite set of values with well-defined addition and multiplication operations.

Case: n=2, k=3

In this case when using real numbers the only number of solutions not possible is 11. When using a field with 2 elements 9, 11, 14, 15, and $17-20$ solutions are not possible. If the field has 3 elements 11, 19, or 20 solutions are not possible. Finally, if the field has 5 elements 11 is again the only number of solutions not possible.

Case: n=2, k=4

We have not finished computing everything for this case, but we have finished a portion of it. If the field has two elements $22, 23, 30, 35, 38, 39, 41-43, 46, 47, 49-55$ and $57-70$ solutions are not possible. By adding another element to the field the numbers of solutions not possible is trimmed down to $30, 49-55$ and $57-70$. We know when using real numbers that all numbers of solutions between 1 and 70 are possible, so we are curious to find out how many elements the finite field must have to have the same result as the real numbers.

6. Where Next?

We are currently computing which numbers of solutions are possible using finite fields for a variety of k's, n's, and field sizes. We plan to analyze this information and use it to determine the rules governing these equations using elements from finite fields. Patterns from the finite field case may also lead us to patterns in the original case, from which we can answer some of our original three questions.

Currently, we have a few computers running calculations for $n = 4, k = 2$ and $n = 4, k = 3$. It will take a while for us to get all the necessary results as for even the smaller values of k and n there are trillions of combinations to compute. The number of combinations increases extremely quickly as n and k increase.

Once we have answers for those three primary questions, we hope to use those answers to find a general answer to the question "What numbers of solutions are possible for degree n polynomial equations with $k \times k$ matrices?"

References

[1] Colleen Duffy and Kaitlyn Hellenbrand. Polynomial equations over matrices. UW-Eau Claire.

[2] Dorian Mueller. Prescribed sets of roots of polynomial equations over $M_k(\mathbb{C})$. 2008 REU at Rutgers University.

[3] Marla Slusky. Zeros of 2×2 matrix polynomials.

[4] Robert Lee Wilson. Polynomial equations over matrices. Rutgers University.

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