1. What is the Tensor Coalgebra?

A vector space $V$ over a field $K$ is said to be $\mathbb{Z}_2$-graded if it is equipped with a fixed decomposition $V = V_0 \oplus V_1$. The tensor algebra $T(V)$ is the $\mathbb{Z}_2$-graded algebra $T(V) = \bigoplus_{i=0}^{\infty} V^i$, where $V^0$ is the $K$-basis tensor product of $V$ with itself.

The coalgebra structure on $T(V)$ is the map $\Delta: T(V) \to T(V) \otimes T(V)$ given by

$$\Delta(v_1 \cdots v_n) = \sum_{i=0}^{n} \sum_{\rho} v_{\rho(1)} \cdots v_{\rho(i)} \otimes v_{\rho(i+1)} \cdots v_{\rho(n)}.$$ 

An automorphism of the tensor coalgebra is a bijective map $g: T(V) \to T(V)$ such that $(g \otimes g) \circ \Delta = \Delta \circ g$.

A co-derivation of the tensor algebra is a map $\delta: T(V) \to T(V)$ such that $\delta(1) = 0$ and $\delta(v_1 \cdots v_n) = \delta(v_1) \cdots \delta(v_n)$.

The $\mathbb{Z}_2$-graded Lie bracket of two co-derivations $\delta$ and $\mu$ is given by

$$[\delta, \mu] = \delta \mu - (-1)^{\delta \mu} \mu \delta.$$

4. The moduli space of $A_\infty$ algebras

An invertible even linear map $\lambda: V \to V$ extends in a natural way to a coalgebra automorphism of $T(V)$. Moreover, if $\mu \in C^1(V)$ for $k > 1$, then $\exp(\mu)$ is always defined. An arbitrary coalgebra automorphism $\phi$ can be written in the form $\phi = \exp(\mu_2) \exp(\mu_1) \cdots$, where $\lambda = \exp(\mu_1)$ and $\mu_2 \in C^1(V)$.

$$\phi = \prod_{i=1}^{\infty} \exp(-\lambda^{i-1}) \lambda^i.$$ 

The important fact about the above formula is that it is computable!

We say that $\delta$ and $\mu$ are equivalent $A_\infty$ algebra structures if there is a coalgebra automorphism $\pi$ of the tensor coalgebra such that $\phi(\pi) = \delta$, and write $\delta \sim \pi$.

Theorem 1 Suppose that

$$d = d_1 + d_{k+1} + \cdots$$

is a differential. Then

$$0 = [d, d] = \sum_{n \geq 0} [d, d_n] + [d_{k+1}, d_n] + \cdots + [d_{k+1}, d_{k+1}]$$

Define $D_0(\phi) = [d, \phi]$. Then the above equation yields an infinite sequence of equations

$$D_0(\phi) = \sum_{n \geq 0} [d, d_n] + k - k.$$ 

This means we can solve for $d_n$ in terms of the operator $D_0$ and $d_{k+1}, \ldots, d_{k+1}$. In particular, if $r$ is the first coefficient larger than $k$ such that $d_r \neq 0$, then $D_0(\delta) = 0$. Suppose that $d_r = D_0(\delta)$. Then $d = (\exp(-\lambda)) \delta$

$$[d, d] = \lambda + \lambda.$$ 

where $\lambda$ stands for terms of higher order than $r$. Since $d_r = [d, \lambda]$, the terms of order $r$ cancel, and we are left with an equivalent co-derivation of the second term of higher order than $r$.

Theorem 2 Suppose $H^0(d_k) = 0$ whenever $k > k$. Then any co-derivation $d$ with loading term $d_k$ is equivalent to $d_k$.

5. Extending a co-derifferential

Let $d = d_k + d_{k+1} + \cdots$ be a co-derivation. Then

$$0 = [d, d] = \sum_{n \geq 0} [d, d_n] + [d_{k+1}, d_n] + \cdots + [d_{k+1}, d_{k+1}]$$

Define $D_0(\phi) = [d, \phi]$. Then the above equation yields an infinite sequence of equations

$$D_0(\phi) = \sum_{n \geq 0} [d, d_n] + k - k.$$ 

This means we can solve for $d_n$ in terms of the operator $D_0$ and $d_{k+1}, \ldots, d_{k+1}$. In particular, if $r$ is the first coefficient larger than $k$ such that $d_r \neq 0$, then $D_0(\delta) = 0$. Suppose that $d_r = D_0(\delta)$. Then $d = (\exp(-\lambda)) \delta$

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Theorem 2 Suppose $H^0(d_k) = 0$ whenever $k > k$. Then any co-derivation $d$ with loading term $d_k$ is equivalent to $d_k$.

6. Applying the Theory

If $V = (v_1, \ldots, v_n)$, then a $k$-multi-index $I_k = (i_1, \ldots, i_k)$ is an ordered $k$-tuple of integers between 1 and $n$. If $v_I$ denotes the element $v_{i_1} \cdots v_{i_k} \in V^k$, then define $v_I^0: V^k \to V$ by $v_I^0(v) = (k-1) v_I v^\prime$. Then the $v_I^0$ form a basis for the co-derivations of $T(V)$.

Example 1 As an example, suppose that $V$ is a 1-dimensional odd space. Then for each $k$ there is only 1 multi-index $I_k = (1, \ldots, 1)$. Define $v_{I_k} = v_{I_k}^0$ and $v_{I_k} = v_{I_k}^0$. Then $v_{I_k}$ is odd and $v_{I_k}$ is even. We have the following formulas for the bracket.

$$[v_{I_k}, v_{I_k}] = 0 \quad \text{and} \quad [v_{I_k}, v_{I_k}] = (2k-1) v_{I_k}.$$ 

As a consequence, if $d = d_{I_k} + d_{I_k} + \cdots$, then $d$ is a co-derivation. Moreover, if $d_k = v_{I_k}^0$ is the loading term in $d$, then the above formulas show that $H^0(d_k) = 0$ if $n > 2k$, by Theorem 2. Thus $d_k$ is a coderivation. This means that the moduls space of $A_\infty$ algebras on a 1-dimensional space is a 1 parameter family $d_k \sim v_{I_k}$, indexed by the natural numbers. When we study the deformations on the moduls space, we obtain the following picture, which completely characterizes the deformations of the moduls space.

References


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