Avoiding two elements of $S_3 \wr C_2$ in $S_n \wr C_k$

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1. Preliminaries

The Symmetric Group

The *symmetric group* S_n is the set of permutations of the numbers $1, 2, \ldots, n$. We can think of the symmetric group as functions with domain and range the numbers $1, 2 \ldots, n$. For example if we have permutation $\tau = (4, 3, 5, 1, 2)$, then we can visualize τ as,

$$\begin{array}{cccccc} & 1 & 2 & 3 & 4 & 5 \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \tau = & 4 & 3 & 5 & 1 & 2 \end{array}$$

Let $\varphi \in S_m$ and $\psi \in S_n$ then we say that ψ ***contains*** φ, and write $\varphi < \psi$, if there is a subpermutation in ψ which has the same relative order as φ. In this case we say the subsequence is ***order isomorphic*** to φ. For example if $\varphi = (2, 1, 3)$ and $\psi = (3, 2, 1, 4, 5)$ then the colored numbers in the equation represent two order isomorphic subsequences.

$$(3, 2, 1, 4, 5) \qquad \text{and} \qquad (3, 2, 1, 4, 5)$$

Note that it is possible for two permutations to not contain one another. For example if $\varphi = (1, 2, 3)$ and $\psi = (5, 4, 3, 2, 1)$ then it is clear that no subpermutation of ψ is order isomorphic to φ. If φ is not contained in ψ then we say φ ***avoids*** ψ.

2. Wreath Product Structure

Using the symmetric group and the cyclic group $C_k = \{1, 2, \ldots, k\}$, we define a ***wreath product*** structure. We write this as $S_n \wr C_k$. This elements of this group are permutations in S_n where each number in the permutation is colored by a number between 1 and k. For example,

$$(3^{(2)}, 1^{(1)}, 2^{(2)}) \in S_3 \wr C_2$$

Using this new structure we can extend our notion of containment. We say ψ ***contains*** φ if there is a subsequence of ψ that is order isomorphic to φ and has identical colors. If we extend the previous example we will have $\varphi = (2^{(2)}, 1^{(1)}, 3^{(2)})$ and $\psi = (3^{(2)}, 2^{(1)}, 1^{(2)}, 4^{(2)}, 5^{(1)})$ then the red numbers below represent a containment, but the blue number do *not*,

$$(3^{(2)}, 2^{(1)}, 1^{(2)}, 4^{(2)}, 5^{(1)}) \qquad (3^{(2)}, 2^{(1)}, 1^{(2)}, 4^{(2)}, 5^{(1)})$$

It is also natural to extend avoidance. Note that for a σ to avoid two patterns τ and φ we must have σ avoids τ and σ avoids φ.

We say two elements, τ and φ, are ***Wilf equivalent***, or are in the same *Wilf class*, if τ and φ are avoided by the same number elements in $S_n \wr C_k$ for all n and k.

Statement of Problem

How many Wilf classes are there for two element combinations of elements in $S_3 \wr C_2$?

3. Bijections

To prove that two patterns is $S_3 \wr C_2$ are Wilf equivalent we must find a *bijection* from one to the other. A *bijection* is a function which has exactly one output value for each input. For our problem we have discovered 4 such bijections. The bijections below take $S_n \wr C_k$ into $S_n \wr C_k$ and preserve Wilf equivalence, or $f(\text{an element avoiding } (\mu, \delta)) = \text{an element avoiding } (\gamma, \nu)$.

Permutation Compliment (PC)

$$\mathrm{PC} : (\alpha_1^{(\beta_1)}, \alpha_2^{(\beta_2)}, \ldots, \alpha_n^{(\beta_n)}) \mapsto (n - \alpha_1 + 1^{(\beta_1)}, \ldots, n - \alpha_n + 1^{(\beta_n)})$$

Color Complement (CC)

$$\mathrm{CC} : (\alpha_1^{(\beta_1)}, \alpha_2^{(\beta_2)}, \ldots, \alpha_n^{(\beta_n)}) \mapsto (\alpha_1^{(k-\beta_1+1)}, \ldots, \alpha_n^{(k-\beta_n+1)})$$

Reverse (R)

$$\mathrm{R} : (\alpha_1^{(\beta_1)}, \alpha_2^{(\beta_2)}, \ldots, \alpha_n^{(\beta_n)}) \mapsto (\alpha_n^{(\beta_n)}, , \ldots, \alpha_1^{(\beta_1)})$$

Inverse (Inv)

$$\mathrm{Inv} : (\alpha_1^{(\beta_1)}, \alpha_2^{(\beta_2)}, \ldots, \alpha_n^{(\beta_n)}) \mapsto (\tau_1^{(\beta_{\alpha_1})}, , \ldots, \tau_n^{(\beta_{\alpha_n})}) \quad \text{where } \tau_{\alpha_i} = i$$

These bijections are tricky to understand; however, with a few examples they become quite clear. Luckily, examples and descriptions of these bijections are given in following frames.

4. Our Bijections

In this frame we give a few examples of the bijections defined in frame 3. Since our problem deals with two element combinations of $S_3 \wr C_2$, our bijections will act on two element combinations of $S_3 \wr C_2$.

Let $\varphi = \{(3^{(1)}, 2^{(1)}, 1^{(1)}); (2^{(1)}, 3^{(2)}, 1^{(1)})\}$, then we have the following.

We begin with Permutation Compliment. This bijection shifts the numbers within the permutation according to the value of n.

$$\begin{aligned} \mathrm{PC}(\varphi) =& \{(3 - 3 + 1^{(1)}, 3 - 2 + 1^{(1)}, 3 - 1 + 1^{(1)}); \\ & (3 - 2 + 1^{(1)}, 3 - 3 + 1^{(2)}, 3 - 1 + 1^{(1)})\} \\ =& \{(1^{(1)}, 2^{(1)}, 3^{(1)}); (2^{(1)}, 1^{(2)}, 3^{(1)})\} \end{aligned}$$

Perhaps the easiest to see is Reversal which simply reverses the permutations.

$$\mathrm{R}(\varphi) = \{(1^{(1)}, 2^{(1)}, 3^{(1)}); (1^{(1)}, 3^{(2)}, 2^{(1)})\}$$

Next is Color Compliment. This is similar to Permutation Compliment, except with the colors.

$$\begin{aligned} \mathrm{CC}(\varphi) =& \{(3^{(2-1+1)}, 2^{(2-1+1)}, 1^{(2-1+1)}); \\ & (2^{(2-1+1)}, 3^{(2-2+1)}, 1^{(2-1+1)})\} \\ & \{(3^{(2)}, 2^{(2)}, 1^{(2)}); (2^{(2)}, 3^{(1)}, 1^{(2)})\} \end{aligned}$$

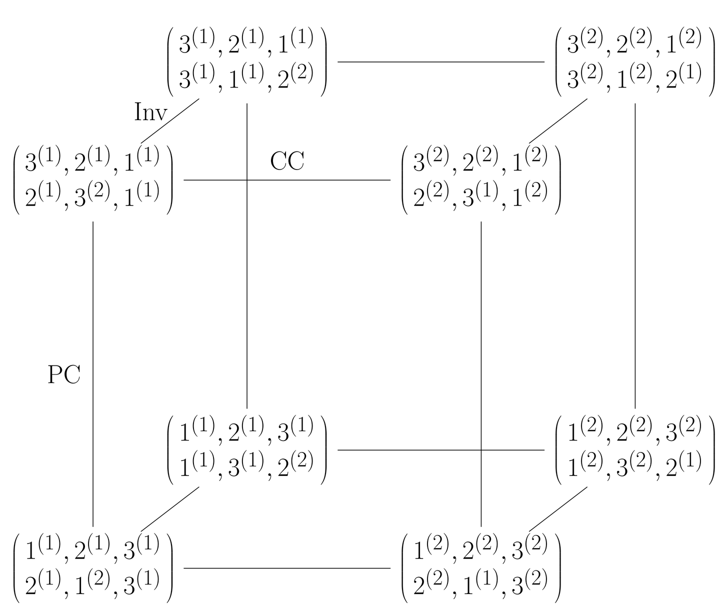
Finally, Inverse. This is the trickiest to see, and is easiest using a diagram similar to the one in Frame 1.

$$\begin{array}{ccc} 1 & 2 & 3 \\ \uparrow & \uparrow & \uparrow \\ 2^{(1)} & 3^{(2)} & 1^{(1)} \end{array}$$

Since the inverse of the first element of φ is itself, this only applies to the second element. Notice how the colors travel with the arrows.

$$\mathrm{Inv}(\varphi) = \{(3^{(1)}, 2^{(1)}, 1^{(1)}); (3^{(1)}, 1^{(1)}, 2^{(2)})\}$$

To visualize the connections created by these bijections we create graphs. Note that Reversal has been omitted since the graph is too complex.



5. Explicit Formula

While bijections will indicate when two patterns are Wilf equivalent, they tell us nothing about a formula for size of the Wilf class. For example, suppose $\varphi = \{(3^{(1)}, 2^{(1)}, 1^{(1)}); (2^{(1)}, 3^{(1)}, 1^{(1)})\}$, then the size of the avoidance of φ is given in the following table,

$k \setminus n$	3	4	5	6
2	46	336	2936	29648
3	160	1864	26656	449504
4	382	6032	177976	2742800
5	748	14856	366896	10811936
6	1294	30928	921016	32792912

Our goal is to find an equation for this table that works for all n and k. We have proved that the following equation works for this table,

$$\sum_{i=0}^{n} (k-1)^{n-i} (n-i)! \binom{n}{i}^2 S(i) \qquad (1)$$

where,

$$S(i) = \begin{cases} 1 & i = 0; \\ 2^{i-1} & i \geq 1 \end{cases}$$

Since equation (1) isn't exactly transparent we include the evaluated sum for $n = 4$ and $n = 5$ respectively.

$$\begin{array}{ll} n = 4 & 24k^4 - 32k + 16 \\ n = 5 & 120k^5 - 400k^2 + 400k - 104. \end{array}$$

Currently our method of generating equations breaks down when more than one index is used in the two permutations. However, with the bijections we automatically have an equation for the entire Wilf class, instead of a single element.

Current Status

In total we have at least 74 Wilf classes. These classes were generated using a computer algorithm which only checks to a low value of n and k. We are searching for additional bijections in order to prove that 74 is an exact number.

In addition, using the procedure outlined here, we have formulas for three Wilf classes, and are attempting to expand these into more general formula for additional Wilf classes.

References

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