Norms in Tensor Products of Banach Spaces

Mitch Phillipson

Faculty Mentor: Simei Tong

University of Wisconsin-Eau Claire

1. Introduction

A function, $||\cdot||$, is said to be a *norm* on a vector space X if $||\cdot||: X \to \mathbb{R}$, satisfies,

1. $||x|| \geq 0$ for all $x \in X$ and $||x|| = 0$ if and only if $x = 0$
2. $||\lambda x|| = |\lambda|\, ||x||$ for all $\lambda \in \mathbb{R}$ and $x \in X$
3. $||x + y|| \leq ||x|| + ||y||$ for all $x, y \in X$

A *Banach Space* is a normed vector space which is complete. By complete we mean every Cauchy sequence converges to a point in the space.

Let $T : X \to Y$ be a bounded operator. Then the norm of T is defined by,

$$\begin{aligned} ||T|| &= \inf\{M : ||Tx||_Y \leq M\, ||x||_X\} \\ &= \sup\{\frac{||Tx||_Y}{||x||_X} : x \neq 0\} \\ &= \sup\{||Tx||_Y : ||x||_X = 1\} \\ &= \sup\{||Tx||_Y : ||x||_X \leq 1\} \end{aligned}$$

Two norms, $||\cdot||_1$ and $||\cdot||_2$, are *equivalent* if there exist positive constants c and C in $\mathbb{R}$ so that for all $x \in X$ we have,

$$c\, ||x||_1 \leq ||x||_2 \leq C\, ||x||_1 .$$

2. Inequalities

Khintchine's Inequality

The *Rademacher's Functions*, $r_n(t)$ on $[0,1]$, are defined by $r_n(t) = \text{sign}(\sin(2^n \pi t))$ for $n \in \mathbb{N}$. For $1 \leq p < \infty$, there exist positive constants A_p and B_p such that for $\{a_n\} \in \ell^2$,

$$A_p \left(\sum |a_n|^2\right)^{\frac{1}{2}} \leq \left(\int_0^1 \left|\sum a_n r_n(t)\right|^p dt\right)^{\frac{1}{p}} \leq B_p \left(\sum |a_n|^2\right)^{\frac{1}{2}}.$$

Rosenthal's Inequality

Let $2 < p < \infty$. If $(x_i)_1^n$ are independent mean zero random variables in L_p, then there exists $K_p < \infty$ so that,

$$\begin{aligned} \frac{1}{2} \max\left\{\left(\sum_{i=1}^n ||x_i||_p^p\right)^{\frac{1}{p}}, \left(\sum_{i=1}^n ||x_i||_2^2\right)^{\frac{1}{2}}\right\} &\leq \left|\left|\sum_{i=1}^n x_i\right|\right|_p \\ &\leq K_p \max\left\{\left(\sum_{i=1}^n ||x_i||_p^p\right)^{\frac{1}{p}}, \left(\sum_{i=1}^n ||x_i||_2^2\right)^{\frac{1}{2}}\right\} \end{aligned}$$

3. Norm given by Patitions and Weights

Define a norm given by partitons and weights as follows: for each partition $P = \{N_i\}$ of $\mathbb{N}$ and function $W : \mathbb{N} \to (0,1]$, then if (a_i) is a sequence of real numbers,

$$||(a_i)||_{P,W} = \left(\sum_i \left(\sum_{j \in \mathbb{N}_i} a_j^2 w_j^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}. \qquad (1)$$

Examples

If $P = \{\{i\} | i \in \mathbb{N}\}$ and $W = (w_n)$ is a sequence of positive numbers, then $X \sim \ell_p$ since

$$||(x_n)|| = ||(x_n)||_{P,W} = \left(\sum_{n=1}^{\infty} (|x_n|^2 w_n^2)^{\frac{p}{2}}\right)^{\frac{1}{p}} = \left(\sum_{n=1}^{\infty} |x_n|^p w_n^p\right)^{\frac{1}{p}}$$

If $P = \{\mathbb{N}\}$ and $W = (w_n)$ is any sequence of positive numbers, then $X \sim \ell_2$ since

$$||(x_n)|| = \left(\left(\sum_{n=1}^{\infty} |x_n|^2 w_n^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}} = \left(\sum_{n=1}^{\infty} |x_n|^2 w_n^2\right)^{\frac{1}{2}}$$

4. Tensor Products

We say a map, f, is a *linear functional* if $f : X \to \mathbb{R}$ and $f(x + y) = f(x) + f(y)$ and $f(ax) = af(x)$ for $x, y \in X$ and $a \in \mathbb{R}$. The space of all linear functionals is denoted $X^{\#}$ and called the *dual* of X. A map, g, is a *bilinear form* if $g : X \times Y \to \mathbb{R}$ and is linear in both coordinants. The space of bilinear forms from $X \times Y$ into $\mathbb{R}$ is denoted $B(X \times Y)$.

The *tensor product*, $X \otimes Y$, of the Banach spaces X and Y can be constructed as a space of linear functionals on $B(X \times Y)$, in the following way: for $x \in X$, $y \in Y$, we denote by $x \otimes y$ the functional given by evaluation at the point (x, y). In other words,

$$(x \otimes y)(A) = A(x, y)$$

We also have the following,

(i) $(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y$

(ii) $x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2$

(iii) $\lambda(x \otimes y) = (\lambda x) \otimes y = x \otimes (\lambda y)$

(iv) $0 \otimes y = x \otimes 0 = 0$.

5. Schectmans Space

In 1975, Schectman created a space called $Y_{p,w} \otimes Y_{p,w}$, which represents a complemented subspace of $L_p(I \times I)$. This space has norm given by,

$$\begin{aligned} ||(a_{i,j})|| = \Bigg[& \sum_{i,j=1}^{\infty} |a_{i,j}|^p + \sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} |a_{i,j}|^2 w_j^2\right)^{\frac{p}{2}} \\ & + \sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} |a_{i,j}|^2 w_i^2\right)^{\frac{p}{2}} + \left(\sum_{i,j=1}^{\infty} |a_{i,j}|^2 w_i^2 w_j^2\right)^{\frac{p}{2}} \Bigg]^{\frac{1}{p}} \end{aligned}$$

It turns out that this space is isomorphic to a space given by four patitions and weights.

$$\begin{aligned} \max\Bigg\{ & \left(\sum_{i,j} |a_{i,j}|^2 w_i^2 w_j'^2\right)^{\frac{1}{2}}, \left(\sum_i \left(\sum_j |a_{i,j}|^2 w_j'^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}, \\ & \left(\sum_j \left(\sum_i |a_{i,j}|^2 w_i^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}, \left(\sum_{i,j} |a_{i,j}|^p\right)^{\frac{1}{p}} \Bigg\} \approx \left|\left|\sum_{i,j} a_{i,j}(x_i \otimes y_j)\right|\right| \end{aligned}$$

6. Results and Open Questions

Theorem: The distance from Y_n to a subspace of L_p goes to ∞ with n, i.e., there is a sequence $(K(n))$, $K(n) \to \infty$, such that for all isomorphisms $T : Y_n \to Z \subset L_p$, $||T||\, ||T^{-1}|| \geq K(n)$

Corollary: The distance between Y_n and $(X_p)^{\otimes n}$ goes to ∞ with n.

Corollary: $(\sum_n Y_n)_{\ell_p}$ with norm given by partitions and weights is not isomorphic to a subspace of L_p.

Proposition: There is a space with a norm given by partitions and weights which has the envelope property, but is not isomorphic to a subspace of L_p.

Open Questions

1. What are necessary and sufficient conditions for a space with norm given by partitions and weights to be isomorphic to a subspace of L_p?
2. Suppose X and Y have norms given by partitions and weights and are each isomorphic to a subspace of L_p. Is $X \otimes Y$ isomorphic to a subspace of L_p?
3. Is there a better constant for $X_p \otimes X_p$ with the envelope norm than what Schectman got by using Rosenthal's inequality twice?

References

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