Chaos and Equicontinuity

Scott Larson

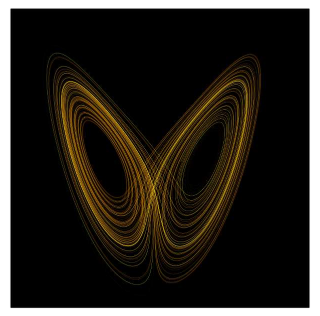
Faculty Mentor: Dr. Don Reynolds

University of Wisconsin-Eau Claire

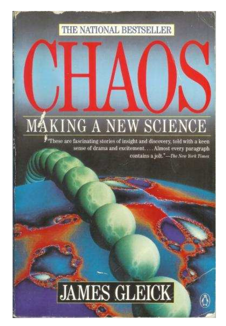
1. History

Chaos Theory

Modern dynamical systems theory began with Henri Poincaré. In 1890 he was studying the three-body problem when he discovered the existence of aperiodic points that approach neither infinity, nor a fixed point. Although this chaotic behavior was observed in 1890, it was not until around 1960 when chaos was formally studied. The invention of the electronic computer made studying chaos possible, by allowing one to iterate a simple function many times. One of the most famous chaotic systems is the Lorenz system, a mathematical model for predicting weather patterns.



The Lorenz Attractor.



Chaos: Making a New Science.

Weather, population, economics, electrical circuits, stability of the solar system, and many other areas of science can all be analyzed using chaos theory. This is the reason why chaotic systems are considered to be one of the important scientific breakthroughs in the twentieth century. In addition to its many applications, chaos has appeared in popular culture in a number of different ways, including James Gleik's best-selling book, "Chaos: Making a New Science." One idea in this book is the butterfly effect, which has become so familiar that a movie was named after it.

2. Chaos Theory

What is Chaos?

The precise definition of chaos is still evolving within the math community. One definition that closely resembles what we would expect a chaotic system to look like is Mario Martelli's definition. Martelli's definition of chaos is equivalent to the following definition:

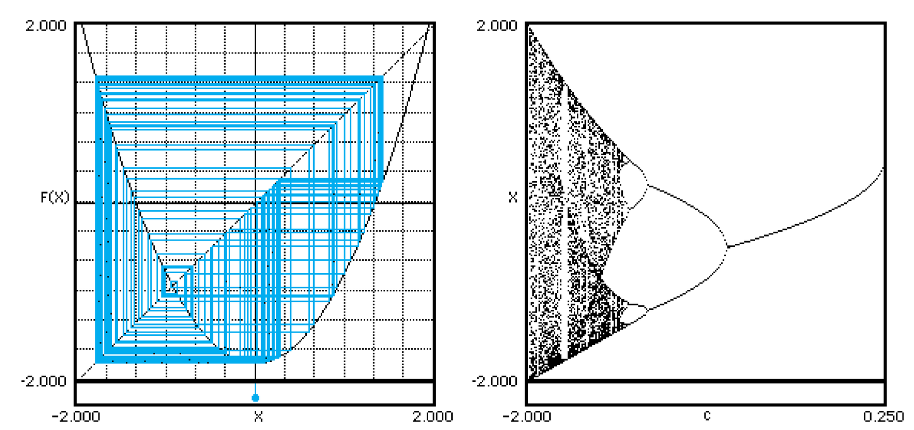
Chaotic System
Let $f : X \to X$ be a continuous function. Then the family of iterates of f is said to be chaotic on X if the following are true:
1. f has sensitive dependence on initial conditions with respect to X.
2. f is topologically transitive in X.

There are a few new definitions involved in this form of chaos that will now be introduced.

Sensitive Dependence on Initial Conditions
$f : X \to X$ has sensitive dependence on initial conditions with respect to X if there exists $\delta > 0$ such that, for any $x \in X$ and any neighborhood $U \subseteq X$ of x, there exists $y \in U$ and $n \geq 1$ such that $|f^n(x) - f^n(y)| > \delta$.

Topologically Transitive
$f : X \to X$ is topologically transitive in X if for any open $U, V \subseteq X$, there is an $n \geq 1$ so that $f^n(U) \cap V$ is non-empty.

A simple example of a chaotic system is obtained by iterating the function $f(x) = x^2 + c$. The following diagrams show the iterates of $f(x)$ for $c = -1.8$ and a bifurcation diagram for different values of c.



The Iterates and Bifurcation Diagram of $f(x) = x^2 + c$.

3. Equicontinuity

When a family of continuous functions act like a single continuous function, the family is said to be equicontinuous. The following is a formal definition of equicontinuity:

Equicontinuous
Let (Y, d) be a metric space. Let $\mathcal{F}$ be a subset of the function space $\mathcal{C}(X, Y)$. If $x_0 \in X$, the set $\mathcal{F}$ of functions is said to be ***equicontinuous at*** x_0 if given $\epsilon > 0$, there is a neighborhood U of x_0 such that for all $x \in U$ and all $f \in \mathcal{F}$,

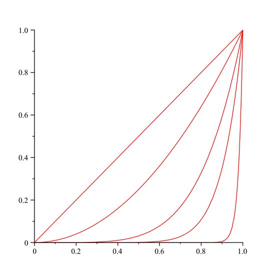
$$d(f(x), f(x_0)) < \epsilon.$$

If the set $\mathcal{F}$ is equicontinuous at x_0 for each $x_0 \in X$, it is said simply to be equicontinuous.

The famous Ascoli and Arzela theorems use equicontinuity to describe when families of continuous functions will converge to a continuous limit function. Such behavior appears to be at the opposite end of the spectrum from chaos theory.

Example

Equicontinuity is best observed through a counter-example. A simple example of a function that is not equicontinuous is $f_n(x) = x^n$ on the interval $[0, 1]$. This family is not equicontinuous at $x = 1$ because for any $x \neq 1$, there is an $\epsilon > 0$ and an $n \geq 1$, so that $d(f(x), f(1)) > \epsilon$. The following is a plot of $f_n(x)$ for different values of n.



$f_n(x) = x^n$ **for** $x \in [0, 1]$**.**

It is easy to see that this family converges to $f(x) = 0$ for $x \in [0, 1)$, and $f(x) = 1$ for $x = 1$, a non-continuous limit function. In fact, it appears that at $x = 1$, this family is sensitive to initial conditions.

4. Connections

Now we are going to find connections between chaos theory and equicontinuity. The previous example is good motivation for us doing this. In fact, we get the following theorem:

Theorem 1
Let $f : X \to X$ be a continuous function. The iterates of f are sensitive to initial conditions if and only if they are not equicontinuous at x for all $x \in X$.

This theorem allows us to connect chaos theory with equicontinuity, and the connections can be extended even further. The following theorem is another way that we can connect chaos theory and equicontinuity.

Theorem 2
Suppose that $f : X \to X$ is topolocially transitive in X. Now suppose that there exists at least one point $x_0 \in X$ that is equicontinuous. Then the set of equicontinuous points is the same as the set of transitivity points.

Now after one more definition, this theorem gives rise to an exciting corollary.

Minimal
Minimal sets are closed nonempty invariant subsets minimal with respect to inclusion.

These sets are important to understand because every closed nonempty invariant set contains a minimal subset. The following is a beautiful corollary to our previous theorem:

Corollary
If $f : X \to X$, where X is minimal, then it either has sensitive dependence on initial conditions or it is equicontinuous.

We will continue to explore the connections between chaos and equicontinuity in the future.

References

[1] Glenn Elert. The Chaos Hypertextbook. http://hypertextbook.com/chaos/

[2] Mario Martelli. Introduction to Discrete Dynamical Systems and Chaos.

[3] S. F. Kolyada Li-Yorke Sensitivity and Other Concepts of Chaos.

[4] Wikipedia. Chaos Theory. 2009. http://en.wikipedia.org/w/index.php?title=Chaos_ theory&oldid=284475721

Acknowledgments

- Dr. Don Reynolds, University of Wisconsin- Eau Claire
- Graphics computed and rendered in LaTeX
- Office of Research and Sponsored Programs, UW-Eau Claire