The Fundamental Theorem of Finite Dimensional Graded Algebras

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1. Introducton

What is an Associative Algebra?

An *associative algebra* is a set with two operations, addition and multiplication, satisfying the following properties:

$$(a+b)+c = a+(b+c) \qquad (ab)c = a(bc)$$
$$a+b = b+a \qquad ab = ba$$
$$a+0 = a$$
$$a+(-a) = 0$$

Distributive Laws

$$a(b+c) = ab+ac$$
$$(a+b)c = ac+bc$$

Some familiar examples of associative algebras are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{K}[x]$ and the algebra $\mathfrak{gl}_n(\mathbb{R})$ of $n \times n$ real matrices.

A vector space V is said to be *graded* if it has a decomposition defined by $V = V_0 \oplus V_1$ where V_0 is the even elements and V_1 is the odd elements.

2. Moduli Space

What is a Moduli Space?

The set of all equivalence classes of associative algebra structures on V is called the *moduli space* of associative algebras on V. Our work involves understanding the moduli spaces of algebras for some low dimensional vector spaces V.

If $V = V_0 \oplus V_1$, where the dimension of the even part V_0 is m and the dimension of the odd part V_1 is n, then we say that V is an $m|n$-dimensional vector space, and the moduli space of algebra structures on V is the moduli space of $m|n$-dimensional algebras.

When our space V is graded, we only consider even maps $g : V \to V$, so the moduli space is the equivalence class of associative algebra structures, up to equivalence by even automorphisms of V.

Our overall goal is to view the structure of the moduli space. This includes studying the deformations to see how the moduli space is pieced together. This however, is far beyond the scope of this poster. In this poster we simply construct the moduli space.

3. Extensions of a Vector Space

What is an Extension?

An extension of an algebra W by an algebra M is represented by a *short exact sequence*

$$0 \to M \to V \to W \to 0.$$

where V is the vector space $V = M \oplus W$, equipped with some algebra structure.

In the language of algebra, we have M is an ideal in V, and $W = V/M$ is the quotient algebra. Why we are interested in this construction is that we want to determine the moduli space of all algebras on V by looking at the moduli spaces of algebras on smaller dimensional spaces.

If V has a proper nontrivial ideal M, then we can use the idea of extensions to express V as an extension of the algebra $W = V/M$ by M. Thus, we also have to understand the case when V has no proper nontrivial ideals. In this case we say V is simple.

4. Introduction to Simple and Division Algebras

There are two classical theorems we have been working on generalizing to $\mathbb{Z}_2$-graded algebras. The radical $\mathfrak{N}$ of an algebra V is the maximal nilpotent ideal. An ideal $\mathfrak{N}$ is called nilpotent if $\mathfrak{N}^n = 0$ for some n. A semisimple algebra S is a direct sum of simple algebras.

Fundamental Theorem of Associative Algebras

Let $\mathfrak{N}$ be the radical of V. Then there is an exact sequence

$$0 \to \mathfrak{N} \to V \to S \to 0$$

of algebras, where S is a semisimple algebra.

Wedderburn's Theorem

If S is a finite dimensional associative algebra, then $S \cong M \otimes D$, where M is a matrix algebra and D is a division algebra.

For ordinary associative algebras, we define D to be a division algebra when every nonzero element is invertible. It turns out that we need to change this definition in the $\mathbb{Z}_2$-graded case. If we say $D = D_0 \oplus D_1$ is a $\mathbb{Z}_2$-graded algebra, then it is a division algebra when every nonzero *homogeneous* element is invertible.

5. Classifying Division Algbras

Theorem

If D is a division algebra over the real numbers, then D is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or the algebra $\mathbb{H}$ of quaternions.

Theorem

Let V_0 be an ordinary associative algebra. Let V_1 be the parity reversion ΠV_0, where this parity reversion is the same space as V_0, but every element is considered to be odd, instead of even. Then there is a unique algebra structure on the double V of V_0, $V = V_0 \oplus V_1$ given by

$$(a,0)\cdot(b,0) = (ab,0)$$
$$(a,0)\cdot(0,b) = (0,ab)$$
$$(0,a)\cdot(b,0) = (0,ab)$$
$$(0,a)\cdot(0,b) = (ab,0).$$

If V_0 is a simple ordinary algebra, then V is a simple $\mathbb{Z}_2$-graded algebra. If V_0 is an ordinary division algebra, then V is a $\mathbb{Z}_2$-graded division algebra.

6. Examples and Explanation

Let V be a real division algebra and $D = V_0$, $M = V_1$.

1. If $D = \mathbb{R}$ then the division algebra is one of the following: $\mathbb{R}$, its double $\mathbb{R} \oplus \Pi\mathbb{R}$ or $\mathbb{C}$, where i is considered to be an odd element.
2. If $D = \mathbb{C}$ then the division algebra is one of the following: $\mathbb{C}$, its double $\mathbb{C} \oplus \Pi\mathbb{C}$, or $\mathbb{H}$, where the generators j and k are taken to be odd.
3. If $D = \mathbb{H}$, then the division algebra is one of the following: $\mathbb{H}$, its double $\mathbb{H} \oplus \Pi\mathbb{H}$, or the two algebras $A = \langle 1, \theta \rangle$ where $\theta^2 = \pm 1$ and $\theta x = \bar{x}\theta$.

From these theorems, we are able to classify all finite simple $\mathbb{Z}_2$-graded associative algebras. Our reason for doing this came out of necessity. We want to generate the entire moduli space for a given vector space. Using extensions, we were able to do most of the work and classify much of the moduli space.

However the parts we missed are also important. If the algebra V does not have a nontrivial proper ideal, then V can not be constructed as an extension, but in this case V is simple, so our recent work outlined in this poster allows us to classify these special cases. Thus, we are now able to completely classify the moduli space of all algebras for a given vector space.

References

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