Introduction to the Busemann-Petty Problem

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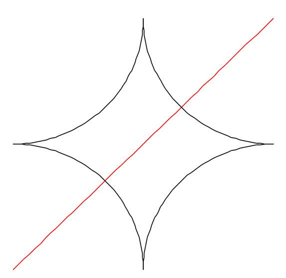
1. Preliminaries

What is a Star Body?

A *Star Body* is a closed and bounded set K in $\mathbb{R}^n$ such that for every $x \in K$ each point in the interval $[0, x)$ is an interior point of K.



In other words a star body has the property that any straight line which passes through the origin crosses the boundary exactly twice. An example is given to the left, notice how the line crosses the boundary twice.

Definition The *Minkowski Functional* is defied as

$$||x||_K = \min\{a \geq 0 | x \in aK\}$$

Definition The radial function of K is defined as

$$\rho_K(x) = ||x||_K^{-1} \quad x \in \mathbb{R}^n$$

If $x \in S^{n-1}$ then ρ_K is the "radius" of K in the x direction.

2. Volume of Sections of Star Bodies

Let χ be the indicator function on the interval $[-1, 1]$, K be a star body in $\mathbb{R}^n$ and M be an m-dimensional subspace of $\mathbb{R}^n$. The volume of the section of K by M can be calculated as follows,

$$\begin{aligned}
\mathrm{Vol_m(K \cap M)} &= \int_M \chi(||x||_K)dx \\
&= \int_{S^{n-1}\cap M} \int_0^{1/||\theta||_K} r^{m-1} dr d\theta \\
&= \frac{1}{m}\int_{S^{n-1}\cap M} \rho_K^m(\theta) d\theta
\end{aligned}$$

Setting $m = n$, then the volume of K is: $\quad \mathrm{Vol_n(K)} = \frac{1}{\mathrm{n}}\int_{S^{n-1}} \rho_{\mathrm{K}}^{\mathrm{n}}(\theta)\mathrm{d}\theta$

Definition (Fourier Transform) We define the *Fourier Transform* of a function $\phi \in L_1(\mathbb{R}^n)$ by

$$\hat{\phi}(u) = \int_{\mathbb{R}^n} \phi(x) e^{-\imath(x,u)} dx, \quad u \in \mathbb{R}^n$$

3. The Busemann-Petty Problem

In 1956 Busemann and Petty posed the famous question which compares the volumes of n-dimensional star bodies by considering volumes of $(n-1)$-dimensional cross sections.

Definition Let $u \in S^{n-1}$, we define $u^\perp$ as the $(n-1)$-dimensional hyperplane perpendicular to u and through the origin.

Problem 1 (The Busemann-Petty Problem) *Suppose K and L are origin symmetric convex bodies in $\mathbb{R}^n$ so that*

$$\mathrm{Vol_{n-1}(K \cap u^\perp) \leq Vol_{n-1}(L \cap u^\perp) \quad for\ all\ u \in S^{n-1}}$$

does it follow that

$$\mathrm{Vol_n(K) \leq Vol_n(L)?}$$

For $n = 2$ the problem asks: If the radius of K is less than the radius of L in all directions, is the area of K less than L? The answer is clearly true since the condition implies that $K \subset L$. The answer has also been proved affirmative for $n = 3, 4$. However, when $n \geq 5$ the answer is negative. This result implies that our **standard three dimensional intuition can not always be extended to higher dimensions.**

4. Intersection Bodies and The Spherical Radon Transform

Definition (Intersection Body) K is the intersection body of L if the radius of K in every direction is equal to the $(n-1)$-dimensional volume of the section of L by the central hyperplane perpendicular to this direction, or for every $u \in S^{n-1}$

$$\rho_K(u) = \mathrm{Vol_{n-1}(L \cap u^\perp)}$$

Definition The *Spherical Radon Transform* R:$C(S^{n-1}) \to C(S^{n-1})$ is a linear operator defined by

$$Rf(u) = \int_{S^{n-1}\cap u^\perp} f(x)dx, \quad u \in S^{n-1}$$

for every function $f \in C(S^{n-1})$.

If K is a star body in $\mathbb{R}^n$ and M be a subspace of $\mathbb{R}^m$, then we can express the volume of K by M in terms of the spherical Radon transform.

$$\mathrm{Vol_m(K \cap M)} = \frac{1}{\mathrm{m}} R(\rho_{\mathrm{K}}^{\mathrm{m}}(\cdot))(\mathrm{M}) = \frac{1}{\mathrm{m}}\int_{S^{n-1}\cap \mathrm{M}} \rho_{\mathrm{K}}^{\mathrm{m}}(\theta)\mathrm{d}\theta$$

5. Fourier Analytic Methods used in Solving the Busemann-Petty Problem

Since the Fourier transform is more popular and easier to use than the Radon transform we wish to connect the two. If f is an even homogeneous function of degree $-n+1$ continuous on S^{n-1}, then

$$Rf(u) = \frac{1}{\pi}\hat{f}(u) \quad \text{for every } u \in S^{n-1}$$

Using the spherical Radon transform one can rewrite the condition for an intersection body in the following way:

$$\rho_K(u) = \mathrm{Vol_{n-1}(L \cap u^\perp)} = \frac{1}{n-1}\int_{S^{n-1}\cap u^\perp} \rho_{\mathrm{L}}^{n-1}(\theta)\mathrm{d}\theta = \frac{1}{\mathrm{n}-1}\mathrm{R}(\rho_{\mathrm{L}}^{\mathrm{n}-1}(\cdot))(\mathrm{u}).$$

This means that a star body K is the intersection body of a star body L if and only if the function $\rho_K(\cdot)$ is the spherical Radon transform of a continuous positive function on S^{n-1}.

In 1988 E. Lutwak [4] proved the following result:

Theorem The solution to the Busemann-Petty problem in $\mathbb{R}^n$ is affirmative if and only if every origin-symmetric convex body in $\mathbb{R}^n$ is an intersection body.

For dimensions $n \geq 5$ it was shown in [6] that cylinders are not intersection bodies, disproving the Busemann-Petty problem.

6. Open Questions

The Generalized Busemann-Petty Problem

Problem 2 *Let K and L be origin symmetric convex bodies in $\mathbb{R}^n$ such that*

$$\mathrm{Vol_m(K \cap H) \leq Vol_m(L \cap H)}$$

for all m-dimensional subspaces H in $\mathbb{R}^n$. Does it follow that

$$\mathrm{Vol_n(K) \leq Vol_n(L)?}$$

This is similar to the Busemann-Petty Problem, except lower dimensional cross sections are being considered. This question has a negative answer when $m \geq 4$ and is still open for $m = 2, 3$. Note that when $m = n - 1$ this is the classic Busemann-Petty problem.

The Slicing Problem

Problem 3 *Let K and L be origin symmetric convex bodies in $\mathbb{R}^n$ be such that*

$$\mathrm{Vol_m(K \cap H) \leq Vol_m(L \cap H)}$$

for all central hyperplanes H. Does there exist a constant c, independent of dimension, such that

$$\mathrm{Vol_n(K) \leq cVol_n(L)}$$

The best known lower bound belongs to Bourgain and depends on the dimension as $n^{\frac{-1}{4}}$, up to a logarithmic term. This problem is equivalent to asking if K is an origin symmetric convex body with volume 1 does there exist a hyperplane section whose $(n-1)$-dimensional volume is greater than c.

References

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