Infinity Algebras and their Deformations

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1. Introduction

What is an Infinity Algebra?

An *infinity algebra* is a vector space V over a field $\mathbb{K}$,equipped with a sequence of n-any operations d_n: $V^n \to V$, that is, maps which take n inputs and give one output satisfying the axioms

$$\sum_{k+l=n+1} [d_k, d_l] = 0, \quad n = 1, 2, \dots$$

where $[d_k, d_l]$ represents the Lie bracket of the two operations considered as coderivations.
The first few relations are:

$$[d_1, d_1] = 0$$
$$[d_1, d_2] = 0$$
$$[d_1, d_3] + 1/2[d_2, d_2] = 0.$$

The first relation says d_1 is a differential on A. The second says that d_1 is a derivation of the algebra structure d_2 on A. The third one says d_2 is associative up to a homotopy determined by d_3.

Two infinity algebra structures d and d' on V are isomorphic or equivalent if there is a coalgebra automorphism g of V such that

$$d' = g^{-1} dg.$$

A coalgebra automorphism g can always be expressed in the form $g = \lambda \prod_{k=2}^{\infty} \exp(\alpha_k)$, where λ is an invertible linear map and α_k is an even coderivation of degree k. The set of all equivalence classes of infinity algebra structures on V is called the *moduli space* of infinity algebras on V.

2.Deformations of Infinity Algebras

Infinitesimal Deformations

A 1-parameter *infinitesimal deformation* of an infinity algebra structure d is a coderivation d_t of the form

$$d_t = d + t\psi,$$

where $D(\psi) = 0$, in other words, ψ is a *cocycle* with respect to the *coboundary operator* D, given by $D(\varphi) = [d, \varphi]$. The deformation is called infinitesimal, because the relation $[d_t, d_t] = 0$ requires that $t^2 = 0$, in other words, we only consider the relation up to first order.
Two infinitesimal deformations d_t and d'_t are infinitesimally equivalent if there is an infinitesimal automorphism $g_t = \exp(t\varphi)$ such that $d'_t = g^*_t(d_t)$. Since $g^*_t(d_t) = d_t + tD(\varphi)$, we determine that $d'_t = d + t\psi'$ is infinitesimally equivalent to $d_t = d + t\psi$ precisely when $\psi' = \psi + d(\varphi)$, in other words, ψ and ψ' are *cohomologous*. This means that the infinitesimal deformations of an infinity algebra structure d are completely classified by the *cohomology* of d.

Formal Deformations

A 1-parameter *formal deformation* of d is a coderivation d_t of the form

$$d_t = d + t\psi_1 + t^2\psi_2 + \cdots,$$

such that

$$[d_t, d_t] = \sum_{n=1}^{\infty} t^n \left(2D(\psi_n) + \sum_{k+l=n} [\psi_k, \psi_l] \right) = 0.$$

3. Versal Deformations

The classification of formal deformations can not be given in a simple manner like the classification of infinitesimal deformations. To understand formal deformations, we construct a type of universal deformation, called the *miniversal deformation* of d, which contains information about all possible formal deformations. To create the versal deformation, we need to discuss deformations with a formal base.

Deformations with base A

If A is a commutative, local algebra A, with augmentation map $\epsilon : A \to \mathbb{K}$, then a deformation d_A with *base* A, is an A_∞ algebra structure d_A on $V \otimes A$, such that under the induced map

$$\epsilon^* = 1 \otimes \epsilon : V \otimes A \to V \otimes \mathbb{K} = V,$$

we have $\epsilon^*(d_A) = d$. If the base A satisfies $A = \overleftarrow{\lim}_n A/\mathfrak{m}^n$, then the deformation is called a formal deformation of d. The previous example of a 1-parameter deformation d_t of A fits this definition, with base $A = \mathbb{K}[[t]]$, the ring of formal power series.

A *versal deformation* is a formal deformation d_A with base A, such that if d_B is another formal deformation with base B, then there is a homomorphism of rings $f : A \to B$, such that $f_*(d_A)$ is formally equivalent to d_B.

A *formal equivalence* g_A is an automorphism of $V \otimes A$ such that $\epsilon_* \circ g_A = \epsilon_*$. It can be expressed in the form $g_A = \exp\left(\sum_i \varphi_i \otimes m_i\right)$, where $\phi_i : V \to V$ is linear and $\{\mathfrak{m}_i\}$ is a basis of the maximal ideal $\mathfrak{m}$.

4. Construction of the Miniversal Deformation.

We can give an explicit construction of a versal deformation, based on extending the construction of the *universal infinitesimal deformation* d^{inf}, given by

$$d^{\text{inf}} = d + \sum_i t_i \delta^i,$$

where $\{\delta_i\}$ is a basis of the cohomology $H(d)$. The *miniversal deformation* d^∞ is of the form

$$d^\infty = d + \sum_i t_i \delta^i + \sum_j x_j \gamma^i,$$

where x_j is a formal power series of order 2 in the parameters t_i, and γ^i is a *pre-basis* of the *coboundaries*. In good situations, the required condition $[d^\infty, d^\infty] = 0$ can be solved for the coefficients x_i. We also obtain some relations R_k on the parameters t_i, which must be satisfied in order for d_t to be well defined. These are called the *relations on the base of the formal deformation*, which is the commutative algebra $\mathbb{K}[[t_1, t_2, \dots]]/(R_1, R_2, \dots)$.

For associative and Lie algebras, we have constructed programs which compute the miniversal deformation.

5. Example: the moduli space of 1|1-dimensional algebras.

The moduli space of associative algebra structures on a 1|1-dimensional complex vector space contains exactly 6 elements, which we label as $d_1, \dots, d_6$. To determine how the moduli space is put together, we first compute the cohomology, and then the versal deformations of each element.

Cohomology:

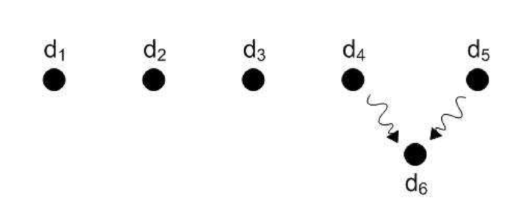
Type	H^0	H^1	H^2	H^3	H^4	H^5	H^6	H^7	H^8
d_1	2	1	1	1	1	1	1	1	1
d_2	0	0	0	0	0	0	0	0	0
d_3	0	0	0	0	0	0	0	0	0
d_4	2	2	2	2	2	2	2	2	2
d_5	1	1	2	2	1	1	2	2	1
d_6	1	0	0	0	0	0	0	0	0

Versal Deformations:

Type	Matrix
d_1^∞	$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
d_2^∞	$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$
d_3^∞	$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
d_4^∞	$A = \begin{bmatrix} 0 & 1 & -1 & 0 \\ t & 0 & 0 & 1 \end{bmatrix}$
d_5^∞	$A = \begin{bmatrix} 0 & t & 0-t & 0 \\ 1 & 0 & 0 & -t \end{bmatrix}$
d_6^∞	$A = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$

6. How the moduli space is glued together

In order to understand the extensions of associative algebras to infinity algebras, we need to compute all of the cohomology. The table of cohomology shows the cohomology for degrees 0 to 8, and suggests the general pattern. We can determine the moduli space of associative algebras completely.



The picture on the left illustrates the manner in which the moduli space is glued together by the miniversal deformations. The miniversal deformations are either equal to the original algebra, or depend only on one parameter. The algebras d_1, d_2 and d_3 have no deformations. The algebra d_4 has a jump deformation to d_6, because $d_t^\infty \sim d_6$ whenever $t \neq 0$. Similarly, d_5 has a jump deformation to d_6. The jump deformations are illustrated by arrows.

References

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