Realizing Keplerian Orbits as Geodesics on a Surface of Revolution

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1. Background

It is well known that in Newtonian physics, the gravitational trajectories in a central gravitational field are conic sections with one focus at the central mass. This is Kepler's First Law. We consider those gravitational trajectories that lie in a fixed plane $\mathbb{P}$ (perhaps thought of as an ecliptic plane), and investigate this question:

To what extent can $\mathbb{P}$ be embedded as a surface of revolution $\mathbb{S}$ inside an abstract three-dimensional Euclidean space $\mathbb{E}$ in such a way that gravitational trajectories are mapped to geodesics on $\mathbb{S} \subset \mathbb{E}$?

We use polar coordinates (ϕ, R) for $\mathbb{P}$, and recall [1] that the differential equation defining the gravitational trajectories is $\left(\frac{d\eta}{d\phi}\right)^2 = \frac{h}{J^2} + \frac{2M}{J^2}\eta - \eta^2$ (†) where $\eta = 1/R$, M is the mass of the object setting up the gravitational field, J is the magnitude of the angular momentum of a particle as it orbits around the central mass, and h is a parameter related to the eccentricity of the orbit. In fact $e^2 = 1 + h\frac{J^2}{M}$ where e is the eccentricity. The general solution to this differential equation is

$$\eta = \frac{M}{J^2}\big(1 + e\cos(\phi - \phi_0)\big). \qquad (1)$$

This is the polar equation of a conic with focus at the origin and eccentricity e. The directrix of the conic is the line $x = J^2/(Me)$ rotated by an angle ϕ_0. To make our problem well-posed, we will regard not only M but also J as a fixed parameter, and thus we will view the solution (1) as depending on the two parameters ϕ_0 and e (or h).

2. Geometry of a Surface of Revolution in $\mathbb{E}$

The space $\mathbb{E}^{3,0}$ is most familiar to our intuition. Sets of points S_c for which $\langle \mathbf{v}, \mathbf{v}\rangle = c$ are round spheres, and the constant c must of course be nonnegative. The space $\mathbb{E}^{2,1}$ is less familiar, although it is reminiscent of Minkowski space. In this case the constant c can be negative, and we find that S_c is a hyperboloid of one-sheet, a cone, or a hyperboloid of two-sheets depending upon whether $c > 0$, $c = 0$ or $c < 0$. We shall find that we can embed $\mathbb{P}$ into $\mathbb{E}$ when $\mathbb{E} = \mathbb{E}^{2,1}$ using the hyperboloids corresponding to $c = \pm J^2/M$.

The differential geometry of a surface of revolution $\mathbb{S} \subset \mathbb{E}$ is determined by its so-called *first fundamental form* $ds^2_{\mathbb{S}}$, which in turn is determined by restricting ds^2 from the ambient space $\mathbb{E}$. If (ϕ, η) are coordinates on $\mathbb{P}$ so that we can describe the map $\mathbb{P} \to \mathbb{E}$ by

$$\mathbf{X}(\phi, \eta) = \Big(x(\phi,\eta), y(\phi,\eta), z(\eta)\Big) = \Big(\alpha(\eta)\cos(\phi), \alpha(\eta)\sin(\phi), z(\eta)\Big) \qquad (2)$$

for some functions $\alpha = \alpha(\eta)$ and $z = z(\eta)$, then $ds^2_{\mathbb{S}} = E(\eta)d\phi^2 + G(\eta)d\eta^2$ where

$$E = \left\langle \frac{\partial \mathbf{X}}{\partial \phi}, \frac{\partial \mathbf{X}}{\partial \phi}\right\rangle \text{ and } G = \left\langle \frac{\partial \mathbf{X}}{\partial \eta}, \frac{\partial \mathbf{X}}{\partial \eta}\right\rangle. \qquad (3)$$

Note that in our (ϕ, η) coordinate system for $\mathbb{S}$, $F = \langle \mathbf{X}_\eta, \mathbf{X}_\phi\rangle = 0$ and E and G depend only on η. Such a coordinate system is said to be an *η-Clairaut patch*.

3. Geodesics on $\mathbb{S}$

Let $\gamma = \gamma(\tau)$ be a geodesic on $\mathbb{S}$. Using our parameterization (2) for $\mathbb{S}$, we can represent γ by $\gamma(\tau) = \langle \alpha(\eta)\cos(\phi), \alpha(\eta)\sin(\phi), z(\eta)\rangle$ where $\eta = \eta(\tau)$ and $\phi = \phi(\tau)$. It is well known [2] that the equations for a geodesic curve in this situation reduce to

$$\phi'' + \frac{E_\eta}{E}\phi'\eta' = 0, \;\; \eta'' - \frac{E_\eta}{2G} + \frac{G_\eta}{2G}\eta'^2 = 0 \qquad (4)$$

where ϕ' and η' denote differentiation with respect to τ.

Now $(E\phi')' = E'\phi' + E\phi'' = (E_\eta\eta' + E_\phi\phi')\phi' + E\phi'' = 0$ by using $E_\phi = 0$ and (4). Therefore $E\phi' = c$ (‡). By solving $\sigma = \langle \gamma', \gamma'\rangle = E\phi'^2 + G\eta'^2 = \frac{c^2}{E} + G\eta'^2$ for η'^2 we get

$$\eta'^2 = \frac{\sigma E - c^2}{EG}. \qquad (5)$$

Thus if we change the parameter of the geodesic γ from τ to ϕ, then we are led to

$$\left(\frac{d\eta}{d\phi}\right)^2 = \frac{\sigma E^2}{c^2 G} - \frac{E}{G} \qquad (6)$$

where c is a constant. This follows from (‡), (5) and the fact that $\dfrac{d\eta}{d\phi} = \dfrac{d\eta/d\tau}{d\phi/d\tau}$.

4. Determining the Surface $\mathbb{S}$

In order to determine the first fundamental form of $\mathbb{S}$ we must determine the functions E and G. To do this, we compare (†) and (6). This leads us to the three equations: $\frac{E^2}{G} = \lambda =$ constant, $\frac{h}{J^2} = \frac{\sigma\lambda}{c^2}$, and $-\frac{E}{G} = \frac{2M\eta}{J^2} - \eta^2$. This determines E and G:

$$E(\eta) = \frac{J^2\lambda}{J^2\eta^2 - 2M\eta}, G(\eta) = \frac{J^4\lambda}{(J^2\eta^2 - 2M\eta)^2}$$

Now in order to determine $\mathbb{S}$, we determine the functions α and z in (2) by using (3). We find

$$\alpha(\eta) = \sqrt{\frac{J^2\lambda}{J^2\eta^2 - 2M\eta}}$$

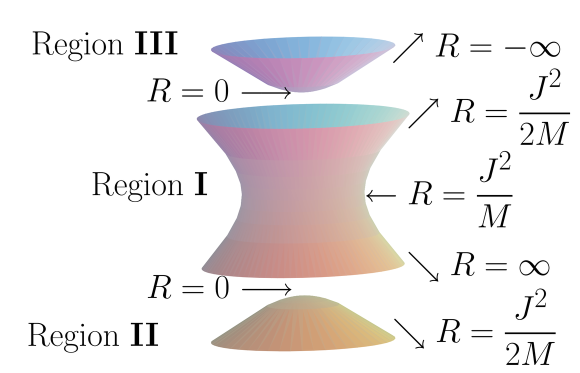
$$z_{30}(\eta) = \int \sqrt{G(\eta) - \alpha'(\eta)^2}\, d\eta, \;\; z_{21}(\eta) = \int \sqrt{\alpha'(\eta)^2 - G(\eta)}\, d\eta$$

where we use z_{30} or z_{21} depending on whether $\mathbb{S}$ is in $\mathbb{E}^{3,0}$ or $\mathbb{E}^{2,1}$.

In order for real solutions $\alpha(\eta)$ and $z(\eta)$ to exist, we must have $E(\eta) > 0$ and $G(\eta) - \alpha'(\eta)^2 > 0$ (for z_{30}) or $\alpha'(\eta)^2 - G(\eta) > 0$ (for z_{21}). Since $G(\eta) - \alpha'(\eta)^2 = \dfrac{\lambda J^2 M^2}{\eta^3(2M - J^2\eta)^3}$ we find that no solutions exist for the case of $\mathbb{E}^{3,0}$ but that solutions exist for $\mathbb{E}^{2,1}$.

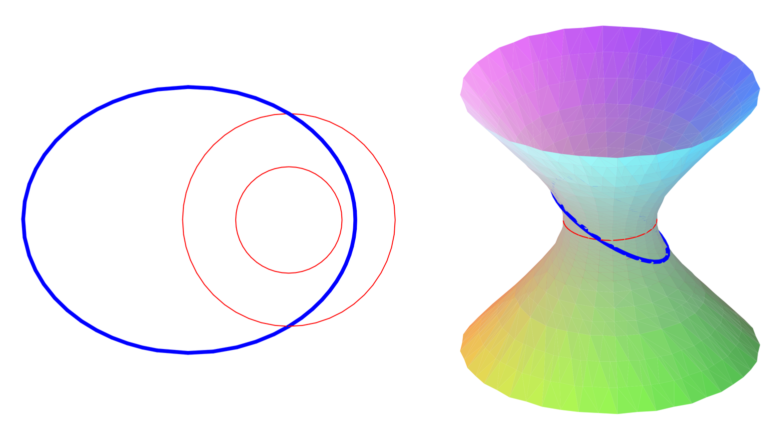
5. The Correspondence Between $\mathbb{P}$ and $\mathbb{S}$

Range for R	λ	$\mathbb{S}$	$\sigma = \langle \gamma', \gamma'\rangle$	e	Important Limits
$R > J^2/(2M)$ I	-1	hyperboloid of one sheet	positive or negative	$e < 1$ (ellipses) or $e > 1$ (hyperbolae)	$\lim_{R \to \left(\frac{J^2}{2M}\right)^+} z_{21} = \infty$; $\lim_{R\to\infty} z_{21} = -\infty$
$0 < R < J^2/(2M)$ II	$+1$	hyperboloid of two sheets (lower sheet where $z_{21} < 0$)	positive	$e > 1$ (hyperbolae)	$\lim_{R\to 0^+} z_{21} = -\frac{J^2}{M}$; $\lim_{R \to \left(\frac{J^2}{2M}\right)^-} z_{21} = -\infty$
$R < 0$ III	$+1$	hyperboloid of two sheets (upper sheet where $z_{21} > 0$)	positive	$e > 1$ (hyperbolae)	$\lim_{R\to 0^-} z_{21} = \frac{J^2}{M}$; $\lim_{R\to -\infty} z_{21} = \infty$

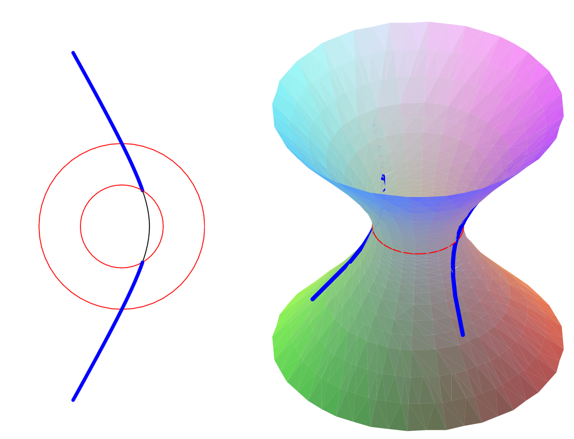


We define $\mathbb{S}$ to be the surface defined by equation $\alpha(\eta)^2 - z_{21}(\eta)^2 = -\frac{\lambda J^4}{M^2}$ for $\lambda = \pm 1$. Thus $\mathbb{S}$ consists of a hyperboloid of one-sheet and a hyperboloid of two-sheets. We find that elliptical trajectories map to geodesics on the hyperboloid of one-sheet. Furthermore we find that the branch of a hyperbolic trajectory with $R > 0$ maps partially to the hyperboloid of one-sheet and partially to the lower-branch of the hyperboloid of two sheets. Branches of hyperbolic trajectories with $R < 0$ correspond to geodesics on the upper sheet of the hyperboloid of two-sheets. These results are visible in the table above.

6. Visualizing Geodesics on $\mathbb{S}$



A elliptical Keplerian orbit and the corresponding geodesic on $\mathbb{S}$. The circles on the right are the circles $R = J^2/(2M)$ and $R = J^2/M$. The latter corresponds to the narrow waist of $\mathbb{S}$, which is obviously a geodesic.



A hyperbolic Keplerian orbit and the corresponding geodesic on $\mathbb{S}$. The circles on the right are the circles $R = \frac{1}{2}\frac{J^2}{M}$ and $R = \frac{J^2}{M}$. On this branch of the hyperbola, $R > 0$. Only the portion of this branch that lies outside the inner circle maps to a geodesic on the hyperboloid of one sheet. The portion of the branch inside the inner circle maps to a geodesic on the lower branch of the hyperboloid of two sheets. The other branch of the hyperbola corresponds to $R < 0$, and corresponds to a geodesic on the upper sheet of the hyperboloid of two sheets.

References

[1] J. Danby. *Fundamentals of Celestial Mechanics.* Willmann-Bell Inc., Richmond, second edition, 1988.

[2] G. Gray. *Modern Differential Geometry of Curves and Surfaces with Mathematica.* CRC Press, second edition, 1998.

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