list of terms and basic facts; as of 13dec

this is a list of terms you are supposedly familiar with by now.

the real numbers, $\mathbb{R}$; the complex numbers, $\mathbb{C}$; set notation, like $\{t \in \mathbb{R} : t \geq 0\}$; and the other notation on page 1 of the notes.

lists, in particular n-lists, in particular n-vectors; the cartesian product of sets; matrices A, especially written in terms of their columns; the transpose, A^{t}, and the conjugate transpose, A^{c}.

maps, their domain, target, range; $Y^X; = \{f : X \to Y\}$; 1-1, onto; map composition; $(fg)h = f(gh)$; fg 1-1 implies g 1-1; fg onto implies f onto; invertible, inverse, right inverse, left inverse. pigeonhole principle. Use of these terms when discussing the basic problem: given $f : X \to Y$ and $y \in Y$, solve $f(?) = y$; existence, uniqueness of solutions.

vector space, particularly vector space of functions, linear subspace; linear map; $\operatorname{ran} A$, $\operatorname{null} A$ as quite different linear subspaces; $L(X, Y)$; $L(\mathbb{F}^n, \mathbb{F}^m) \sim \mathbb{F}^{m \times n}$; matrix multiplication as map composition. elementary matrices and their inverse.

elimination; bound vs free columns (a column is bound if and only if it is NOT a weighted sum of the columns to the left of it; a matrix is 1-1 if and only if all its columns are bound). rref and rrref; $A = A(:, \texttt{bound})\operatorname{rrref}(A)$. elimination as a means to generate bases for the nullspace and the range of a matrix; also, as a means to solve $A? = y$ and, even, if need be, to construct A^{-1}. If $A \in \mathbb{F}^{m \times n}$, then $m < n$ implies A not 1-1; $m > n$ implies A not onto; $m = n$ implies A 1-1 iff A onto. Invertibility of triangular matrices.

generalization of matrices: column maps; bound vs free columns. basis; basis for $\{0\}$; extending to a basis; thinning to a basis and how to do it; dimension; dimension of $\mathbb{F}^T$; Dimension Formula; $X \subset Y$ implies $\dim X \leq \dim Y$ with equality iff $X = Y$.

$\dim(Y + Z) = \dim Y + \dim Z - \dim(Y \cap Z)$; special case $Y \cap Z = \{0\}$: direct sum decomposition $X = Y \dotplus Z$. (left) inverse of a basis to extract coordinates with respect to that basis. data maps aka row maps; interpolation and linear projectors.

For any matrix $A \in \mathbb{R}^{m \times n}$, $\operatorname{tar} A = \operatorname{ran} A \dotplus \operatorname{null} A^{\mathrm{t}}$.

additional material post-midterm

factorization $A = V\Lambda^{\mathrm{t}}$ is minimal iff V is basis for $\operatorname{ran} A$ iff $\#V = \operatorname{rank} A$ iff Λ is basis for A^{t}.

inner product spaces, especially $\mathbb{F}^n$ with the inner product $\langle x, y\rangle = y^{\mathrm{c}}x$, as a ready source of data maps, namely the maps

$$[w_1, \ldots, w_m]^{\mathrm{c}} : f \mapsto ({w_j}^{\mathrm{c}} f := \langle f, w_j\rangle : j = 1{:}n).$$

Column map V into inner product space Y is 1-1 iff $V^{\mathrm{c}}V$ is 1-1, in which case $P_V := V(V^{\mathrm{c}}V)^{-1}V^{\mathrm{c}}$ is the orthogonal projector onto $\operatorname{ran} V$. $P_V b$ is the least-squares solution to the linear system $V? = b$, i.e., $f = P_V b$ minimizes $\|b - f\|$ over all $f \in ranV$ which is the same as satisfying the corresponding normal equation, $V^{\mathrm{c}}V? = V^{\mathrm{c}}b$. Special case: discrete least-squares approximation. Formula for P_V particularly simple when $V^{\mathrm{c}}V = \mathrm{id}$, i.e., when the columns of V are orthonormal, i.e., form an o.n. basis for $\operatorname{ran} V$. Sufficient to get an orthogonal basis, i.e., V with $V^{c}V$ diagonal (and 1-1). Get such a basis for some linear subspace F from an arbitrary basis W for it by Gram-Schmidt:

$$v_j = w_j - \sum_{k<j} \frac{{v_k}^{c} w_j}{{v_k}^{c} v_k} v_k, \quad j = 1, 2, \ldots .$$

The Euclidean norm, the norm associated with an inner product are examples of a norm (positive definite, absolutely homogeneous, subadditive) on a vector space. The associated map norm $\|A\| := \sup_{x \neq 0} \|Ax\|/\|x\|$ is a norm on $L(X, Y)$ but has the additional important property that $\|AB\| \leq \|A\|\|B\|$ (if AB is defined).

The condition $\kappa(V)$ of a basis and its relation to the relative residual $\|Vx - Va\|/\|b\|$ and the relative error $\|x - a\|/\|x\|$ in an approximation a to the solution of $V? = b$.

similarity *vs* equivalence

Eigenvectors and eigenvalues, spectrum $\operatorname{spec}(A)$, defective *vs* nondefective eigenvalue, eigenbasis, diagonable or not, can split off nondefective eigenvalues; Schur form, matrix is normal (i.e., $A^{c}A = AA^{c}$) iff has o.n. eigenbasis; Hermitian has o.n. eigenbasis and real spectrum. Minimal annihilating polynomial for A at x, constructed via elimination, getting eigenvalues along with eigenvectors IF one can find the zeros of that polynomial.

A is powerbounded iff for all $\mu \in \operatorname{spec}(A)$ $|\mu| \leq 1$ with equality only if μ is not defective.

A is convergent iff for all $\mu \in \operatorname{spec}(A)$ $|\mu| \leq 1$ with equality only if μ is not defective and $\mu = 1$.

A is convergent to 0 iff $\rho(A) < 1$.

Gershgorin circles; characteristic polynomial; geometric *vs* algebraic multiplicity.

this is quite impressive!