

Suboptimal Model Predictive Control (Feasibility Implies Stability)

P. O. M. Scokaert, D. Q. Mayne, and J. B. Rawlings

Abstract— Practical difficulties involved in implementing stabilizing model predictive control laws for nonlinear systems are well known. Stabilizing formulations of the method normally rely on the assumption that global and exact solutions of nonconvex, nonlinear optimization problems are possible in limited computational time. In this paper, we first establish conditions under which suboptimal model predictive control (MPC) controllers are stabilizing; the conditions are mild holding out the hope that many existing controllers remain stabilizing even if optimality is lost. Second, we present and analyze two suboptimal MPC schemes that are guaranteed to be stabilizing, provided an initial feasible solution is available and for which the computational requirements are more reasonable.

Index Terms— Dual-mode control, nonconvex nonlinear optimization, nonlinear model predictive control, suboptimal control.

I. INTRODUCTION

Conventional formulations of nonlinear model predictive control (MPC) require, at each sampling instant, an exact global solution of a nonconvex, nonlinear program. To ensure stability the nonlinear program includes a stability constraint, normally an equality constraint on the terminal state [1], [2]. This makes the implementation of stabilizing MPC difficult on at least two counts. On the one hand, exact satisfaction of nonlinear equality constraints cannot, in general, be achieved in finite computational time and early termination of the optimization may cause errors that affect stability. On the other hand, global solutions cannot usually be guaranteed, or are highly computationally expensive. The theory behind nonlinear MPC is consequently often inapplicable, although in some applications it may be possible to employ global optimization. This has been done in the context of specific control applications [3], but not yet in MPC.

To reduce problems associated with the terminal equality constraint $x(k+N) = 0$, one proposal [4] (for continuous time systems) replaces this constraint by an inequality constraint $x(k+N) \in W$ and employs a local asymptotically stabilizing controller $h_L(\cdot)$ in W ; the set W is required, *inter alia*, to be positively invariant under $h_L(\cdot)$. Another interesting version of MPC employs infinite horizon cost and finite horizon control [5]–[7], an approach normally restricted to linear systems. Interestingly, this approach tests whether the state at the end of the control horizon lies in the output admissible set [8] and varies the control horizon, if necessary, to satisfy this test. Hence, a test implicitly of the form $x(k+N) \in W$ is used.

In an interesting paper [9] (which appeared after this paper was submitted), the authors identify difficulties in transposing the continuous-time results of [4] to discrete-time MPC and propose a fixed horizon dual-mode (optimal) MPC strategy for linear time invariant systems.

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Their dual-mode controller employs a terminal constraint of the form $x(k+N) \in D \subset D \subset W$ and a stage cost that is zero in D , both assumptions being similar to, if somewhat more complex than, ours (see, e.g., Assumption A2). Theorem 2 (below), which establishes stability of fixed-horizon dual-mode optimal MPC is similar if the system is linear to [9, Theorem 1].

To reduce the severe computational problems associated with nonconvexity, a *suboptimal* approach for continuous-time systems, proposed in [4] and discussed informally in [10] and [11], employs an initial feasible solution which is improved iteratively in lieu of optimization. A variable horizon strategy was employed. The purpose of this paper is to extend these results by showing that under mild conditions, feasibility rather than optimality is sufficient for stability and to establish stability of suboptimal fixed horizon versions of MPC for nonlinear discrete-time systems. We examine two cases: first, when the stability constraint is $x_{k+N} = 0$ and second, when it is $x_{k+N} \in W$; we then compare the results with optimal MPC employing these two stability constraints.

The paper is organized as follows. Section II sets up the background and notation for the paper. Here, we present a result that establishes that feasibility, rather than optimality, is sufficient for stability. In Section III, we briefly review optimal MPC strategies and highlight their stabilizing properties. The suboptimal MPC schemes we propose are discussed in Section IV; their stabilizing properties are established and their computational demands discussed. In Section V, we present some illustrative examples and concluding remarks are made in Section VI.

II. FEASIBILITY IMPLIES STABILITY

We consider discrete-time nonlinear systems described by

$$x_{k+1} = f(x_k, u_k) \quad (1)$$

where $x_k \in \mathbb{R}^n$ and $u_k \in \mathbb{R}^m$ denote the state and control vectors at discrete time k and $f(\cdot): \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is assumed to be continuous at the origin with $f(0, 0) = 0$.

The objective is to regulate the state x to the origin and we consider receding-horizon control laws that determine at each sampling instant k , state x_k , a finite sequence of future controls

$$\pi_k = \{u_{k|k}, u_{k+1|k}, \dots, u_{k+N-1|k}\} \quad (2)$$

to satisfy certain constraints; here N is the control horizon. Let $\{x_{k|k}, x_{k+1|k}, \dots, x_{k+N|k}\}$ where $x_{k|k} = x_k$ denote the corresponding state sequence. The current control action u_k is chosen to be the first vector in the sequence π_k , i.e.,

$$u_k = u_{k|k} \quad (3)$$

for all k . If the control u is a continuous function of the state x , Lyapunov stability theory establishes convenient conditions for asymptotic stability. In suboptimal control, the control employed is not unique and may also vary discontinuously with the state. Hence, the conditions under which suboptimal control is stabilizing needs to be more carefully examined. Indeed, at each sampling instant k , the only demand placed on the control profile π_k is that it belongs to a set Π_k (defined by inequalities); π_k can be an arbitrary element of this set so that many different values of control u_k can be (and are) selected for a given state x_k . In this section, we present a result similar to the standard Lyapunov stability theorem, but which allows nonuniqueness and discontinuity in the control law. This result simplifies subsequent analysis. In the sequel, we use (asymptotic) stability of a system to

mean that the system has an (asymptotically) stable equilibrium at the origin. A function $\alpha(\cdot)$, defined on nonnegative reals, is a K function if it is continuous, strictly increasing with $\alpha(0) = 0$. For all $r \geq 0$, $n \geq 1$, $B_r^n := \{x \in \mathbb{R}^n : \|x\| \leq r\}$.

Theorem 1: Let there exist:

- 1) a function $V(\cdot): \mathbb{R}^n \times \mathbb{R}^{N^m} \rightarrow \mathbb{R}$ continuous at the origin with $V(0, 0) = 0$ and a K -function $\alpha(\cdot)$, such that for all $x \in \mathbb{R}^n$, $\pi \in \mathbb{R}^{N^m}$,

$$V(x, \pi) \geq \alpha(\|x\|) \quad (4)$$

- 2) a set $\mathbb{F} \subseteq \mathbb{R}^n$ that contains an open neighborhood of the origin and a K -function $\gamma(\cdot)$, such that every realization $\{x_k, \pi_k\}$ of the controlled system with $x_0 \in \mathbb{F}$ satisfies $x_k \in \mathbb{F}$ for all $k \geq 0$ and

$$V(x_{k+1}, \pi_{k+1}) - V(x_k, \pi_k) \leq -\gamma(\|(x_k, u_k)\|) \quad (5)$$

with u_k denoting the first element of π_k ;

- 3) a constant $r > 0$ and a K function $\sigma(\cdot)$, such that every realization $\{x_k, \pi_k\}$ of the controlled system with $x_k \in B_r^n$ satisfies

$$\|\pi_k\| \leq \sigma(\|x_k\|). \quad (6)$$

Then, the controlled system is asymptotically stable in \mathbb{F} .

Proof:

- *Stability:* Let $\{x_k\}$ represent a trajectory of the controlled system commencing at an arbitrary point $x_0 \in \mathbb{F}$. Because V is continuous at the origin, with $V(0, 0) = 0$, there exist a constant $r_1 > 0$ and a K function $\beta(\cdot)$ such that $V(x, \pi) \leq \beta(\|(x, \pi)\|)$ for all $x \in B_{r_1}^n$, $\pi \in B_{r_1}^{N^m}$; also, as \mathbb{F} contains the origin in its interior, there exists a constant $r_2 > 0$ such that $B_{r_2}^n \subseteq \mathbb{F}$. For any $\epsilon > 0$, there exists $\delta > 0$ such that: 1) $\delta \leq \min(r, r_1, r_2)$; 2) $\sigma(\delta) \leq r_1$; and 3) $\beta(\delta + \sigma(\delta)) < \alpha(\epsilon)$; $\delta > 0$ exists because $\alpha(\epsilon) > 0$ and $\sigma(\delta) \rightarrow 0$ as $\delta \rightarrow 0$, so that $\beta(\delta + \sigma(\delta)) \rightarrow 0$ as $\delta \rightarrow 0$. Suppose $\|x_0\| \leq \delta$; then $\|\pi_0\| \leq \sigma(\delta)$ and $V(x_0, \pi_0) \leq \beta(\|(x_0, \pi_0)\|) \leq \beta(\|x_0\| + \|\pi_0\|) \leq \beta(\delta + \sigma(\delta)) < \alpha(\epsilon)$. Moreover, since $\|x_0\| \leq \delta$, we have $V(x_k, \pi_k) \leq V(x_0, \pi_0) < \alpha(\epsilon)$ for all $k \geq 0$. Also, $\alpha(\|x_k\|) \leq V(x_k, \pi_k)$ for all $k \geq 0$. Therefore, we have $\alpha(\|x_k\|) < \alpha(\epsilon)$ and it follows that $\|x_k\| < \epsilon$ for all $k \geq 0$, all $x_0 \in B_\delta^n$.
- *Convergence:* In view of (4), we have $V(x, \pi) \geq 0$ for all x, π . Furthermore, it follows from (5) that V decreases along trajectories of the controlled system that commence in \mathbb{F} . It follows that with $x_0 \in \mathbb{F}$, $V(x_k, \pi_k) \rightarrow V^*$ as $k \rightarrow \infty$, where V^* is a nonnegative constant. We conclude that $V(x_{k+1}, \pi_{k+1}) - V(x_k, \pi_k) \rightarrow 0$ as $k \rightarrow \infty$ and this implies that $\gamma(\|(x_k, u_k)\|) \rightarrow 0$. Because $\gamma(\cdot)$ is a K function, it follows that $x_k, u_k \rightarrow 0$ as $k \rightarrow \infty$. \square

If the initial sequence π_0 is feasible, so, in the nominal case, are all subsequent sequences π_k computed according to

$$\pi(k) = \{(v_{k|k-1}, \dots, v_{k+N-2|k-1}, h_L(x_{k+N-1|k-1}))\}$$

where $\pi_{k-1} = \{v_{k-1|k-1}, v_{k|k-1}, \dots, v_{k+N-1|k-1}\}$. Theorem 1 shows that initial feasibility is sufficient for nominal stability.

III. OPTIMAL MPC STRATEGIES

The goal of MPC is to regulate the state of the system to the origin while satisfying control and state constraints of the form

$$\begin{aligned} u_k &\in \mathcal{U} \\ x_k &\in \mathcal{X} \end{aligned} \quad (7)$$

for all k . Optimal MPC minimizes, at each state-time pair (x, k) , an objective function

$$\phi(x, \pi) = \sum_{j=k}^{k+N-1} L(x_j, v_j) \quad (8)$$

subject to

$$x_{j+1} = f(x_j, v_j), \quad x_k = x \quad (9)$$

the control and state constraints (7) and a stability constraint (see below); here, $\pi = \{v_k, \dots, v_{N-1}\}$ is the decision variable. Because f and L are time invariant, the solution to this optimal control problem is $\pi_k^0(x) = \pi^0(x) = \{\pi_1^0(x), \dots, \pi_N^0(x)\}$; $h(\cdot) := \pi_1^0(\cdot)$ is the implicit MPC control law. The control applied to the plant (at (x, k)) is, therefore

$$u_k = h(x). \quad (10)$$

The resultant value function for the optimal control problem is

$$\phi(x) := V(x, \pi^0(x)). \quad (11)$$

“Classical” fixed horizon MPC (for nonlinear discrete time systems) employs the stability constraint

$$x_{k+N} = 0. \quad (12)$$

In this note, we use classical MPC to denote model predictive control with a terminal equality stability constraint. Dual-mode fixed-horizon MPC, on the other hand, employs the stability constraint

$$x_{k+N} \in W \quad (13)$$

where W is a convex compact subset of \mathbb{X} which contains the origin in its interior. Inside W a locally stabilizing control law $u = h_L(x)$ ($h_L(\cdot): W \rightarrow \mathcal{U}$) satisfying

$$L(x, h_L(x)) = 0, \quad \forall x \in W. \quad (14)$$

is employed. When $W = \{0\}$, the local control law is $u = h_L(0) = 0$, in which case this requirement is satisfied if $L(0, 0) = 0$. The set W is chosen to be positively invariant for the system $x_{k+1} = f(x_k, h_L(x_k))$. A method for constructing W and h (presented in [4]) can be extended to the discrete-time case (see the Appendix).

The following assumptions are made:

- A0:** $f(\cdot)$ is continuous at the origin, with $f(0, 0) = 0$;
- A1:** $L(\cdot)$ is continuous at the origin with $L(0, 0) = 0$;
- A2:** $L(x, h_L(x)) = 0$ for all $x \in W$;
- A3:** there exists a K function $\ell(\cdot)$ such that $L(x, u) \geq \ell(\|(x, u)\|)$ for all $x \notin W$ and for all u .

If $W = \{0\}$, Assumption A3 simply requires that $L(x, u) \geq \ell(\|(x, u)\|)$ for all x, u . We may, for instance, choose $L(x, u) = x'Qx + u'Ru$, with Q and R positive definite. If $W \neq \{0\}$, a function which satisfies Assumptions A1–A3 is

$$L(x, u) = \theta(x)\mathcal{L}(x, u) \quad (15)$$

where $\mathcal{L}(x, u) = x'Qx + u'Ru$, with Q and R positive definite and

$$\theta(x) = \begin{cases} 0, & \text{if } x \in W \\ 1, & \text{otherwise.} \end{cases} \quad (16)$$

That optimal fixed-horizon model predictive control with a terminal equality stability constraint is stabilizing is established by Keerthi and Gilbert [1]. Optimal fixed-horizon dual-mode MPC for discrete time linear systems has been analyzed in [9]. For nonlinear systems, we have the following result.

Theorem 2: Let \mathbb{F} denote the set of states for which there exists a control sequence that satisfies (7), (9), and (13). Optimal fixed-horizon dual-mode MPC is asymptotically stabilizing with a region of attraction \mathbb{F} .

Proof: Stability is a local property and follows from the stabilizing properties of the control law $u = h_L(x)$ in W . Also, if the state enters W , convergence to the origin follows from the properties of h_L . It therefore only remains to show that all trajectories of the controlled system, commencing in \mathbb{F} , enter W in finite time.

Suppose $x_0 \notin W$. Because W contains an open neighborhood of the origin, there exists a constant $r > 0$ such that $x \notin W \Rightarrow \|x\| \geq r$. It follows from Assumption A3 that $x \notin W \Rightarrow L(x, u) \geq \ell(r)$ for all u . Since $L(x, h_L(x)) = 0$ for all $x \in W$, it follows that $\phi(f(x, h(x))) - \phi(x) \leq -L(x, h(x))$ for all x so that the optimal value of the objective decreases by at least $L(x_k, u_k)$ at each sampling instant k .

Let \bar{k} denote a finite integer such that $\bar{k}\ell(r) > \phi(x_0)$. If the state has not entered W by time $k = \bar{k}$, we have $\|x_k\| \geq r$ and, therefore, $L(x_k, u_k) \geq \ell(r)$ for $k = 0, 1, \dots, \bar{k}$. It follows that $\phi(x_{\bar{k}}) \leq \phi(x_0) - \bar{k}\ell(r) < 0$ a contradiction since the objective is, by definition, nonnegative. We conclude that $x_0 \in \mathbb{F} \Rightarrow x_{\bar{k}} \in W$ with \bar{k} finite. This completes the proof.

Thus, the dual-mode MPC strategy is, under mild conditions, stabilizing. By casting the stability condition as an inequality, computational problems associated with exact satisfaction of nonlinear equality constraints are reduced. Nevertheless, global solutions to a constrained nonlinear optimal control problem must still be obtained iteratively. Since most optimization methods, when applied to a nonconvex problem, yield local rather than global minima, we turn our attention to suboptimal strategies, which are more easily obtained.

IV. SUBOPTIMAL MPC STRATEGIES

In this section, we propose suboptimal versions of the two MPC laws presented in Section III to illustrate the fact that feasibility is sufficient for stability; in these versions, the computational requirement is reduced to finding a control profile that satisfies the control, state, and stability constraints. The solution need not minimize the objective either globally or locally and, since in the nominal case, recalculation of the control profile at each sampling instant is theoretically not necessary, the previously obtained control profile is normally an excellent ‘‘hot start’’ for the current nonlinear program. Finding a control profile that satisfies a set of constraints is significantly easier than solving a global optimization problem; even this problem is made easier by the availability of ‘‘hot starts’’; indeed, in the nominal case at least, all that is required is an initial feasible control profile.

A suboptimal version of ‘‘classical’’ fixed-horizon MPC (which employs a terminal equality stability constraint) is presented below.

Algorithm 1—Suboptimal Classical MPC:

- Choose $\mu \in (0, 1]$.
- At time $k = 0$, state x_0 , find a control sequence $\pi_0 = \{v_{0|0}, v_{1|0}, \dots, v_{N-1|0}\}$ that satisfies (7), (9), and (12); set $u_0 = v_{0|0}$.
- At time k , state x_k , choose a control sequence $\pi_k = \{v_{k|k}, \dots, v_{k+N-1|k}\}$ that satisfies (7), (9), (12), and $\phi(x_k, \pi_k) \leq \phi(x_{k-1}, \pi_{k-1}) - \mu L(x_{k-1}, u_{k-1})$ using $\pi = \{v_{k|k-1}, \dots, v_{k+N-2|k-1}, 0\}$ as an initial guess. Set $u_k = v_{k|k}$.

In the nominal case, when the model is exact and there are no disturbances, π satisfies all the constraints on π_k , even if $\mu = 1$. This choice for π_k requires no further computation. In practice, however, model inaccuracies and disturbances may cause the control sequence

π not to satisfy (7), (9), and (12) or not to yield a cost reduction of $-L(x_{k-1}, u_{k-1})$ as in the nominal case. Then a new control sequence π_k is computed, which yields the required cost reduction. Small μ values make this requirement easier to achieve.

If the model is very inaccurate or disturbances are large, there may not exist a control sequence that satisfies (7), (9), and (12) and yields a cost reduction. Then the algorithm fails, regardless of μ . The best strategy may then be to give up on cost reduction and to simply find a new control sequence π_k that satisfies (7), (9), and (12) as is done at time $k = 0$.

The stabilizing properties of the control law that arise by implementation of Algorithm 1 are established below.

Theorem 3: Let \mathbb{F} represent the set of states for which there exists a control sequence that satisfies (7), (9), and (12). If there exists a K function, $\sigma(\cdot)$, and a constant $r > 0$ such that $\|\pi_k\| \leq \sigma(\|x_k\|)$ for all $x_k \in B_r^n$, the suboptimal MPC law is asymptotically stabilizing with a region of attraction \mathbb{F} .

Proof: It follows from Assumptions A0 and A1 that the MPC objective ϕ is continuous at the origin, with $\phi(0, 0) = 0$. Furthermore, $\phi(x, \pi) \geq L(x, u)$ for all x, π , where u denotes the first control in the sequence π ; in view of Assumption A3, it follows that $\phi(x, \pi) \geq \ell(\|x, u\|)$, for all x, π , since $W = \{0\}$. The first condition of Theorem 1 is, therefore, satisfied.

The algorithm also ensures that $\phi(x_{k+1}, \pi_{k+1}) - \phi(x_k, \pi_k) \leq -\mu L(x_k, u_k)$ along trajectories of the controlled system. This satisfies the second condition of Theorem 1 since $\mu > 0$ and $L(\cdot)$ is bounded below by a K function (Assumption A3).

Finally, the third condition of Theorem 1 is satisfied by assumption.

Asymptotic stability therefore follows from Theorem 1. \square

The condition $\|\pi_k\| \leq \sigma(\|x_k\|)$ can be incorporated as an additional nonrestrictive constraint in the algorithm. The suboptimal version of fixed-horizon dual-mode control MPC requires, as does the optimal version, a locally asymptotically stabilizing control law $u = h_L(x)$ such that $h: W \rightarrow \mathcal{U}$, where W is a compact subset of \mathbb{X} , contains an open neighborhood of the origin, and is positively invariant for the system $x_{k+1} = f(x_k, h_L(x_k))$. The suboptimal version of dual-mode MPC is defined by the following algorithm.

Algorithm 2—Suboptimal Dual-Mode MPC:

- Choose $\mu \in (0, 1]$.
- At time $k = 0$, state x_0 if $x_0 \in W$ set $u_0 = h_L(x_0)$. Otherwise, find a control sequence $\pi_0 = \{v_{0|0}, v_{1|0}, \dots, v_{N-1|0}\}$ and corresponding state sequence $\{x_0, x_{1|0}, \dots, x_{N|0}\}$ that satisfies (7), (9), and (13); set $u_0 = v_{0|0}$.
- At time k , state x_k if $x_k \in W$ set $u_k = h_L(x_k)$. Otherwise, choose a control sequence $\pi_k = \{v_{k|k}, v_{k+1|k}, \dots, v_{k+N-1|k}\}$ and corresponding state sequence $\{x_k, x_{k+1|k}, \dots, x_{k+N|k}\}$ that satisfies (7), (9), (13), and $\phi(x_k, \pi_k) \leq \phi(x_{k-1}, \pi_{k-1}) - \mu L(x_{k-1}, u_{k-1})$, using $\pi = \{v_{k|k-1}, \dots, v_{k+N-2|k-1}, h_L(x_{k+N-1|k-1})\}$ as an initial guess. Set $u_k = v_{k|k}$.

As in the first algorithm, π satisfies all the constraints on π_k , even if $\mu = 1$, in the nominal case, i.e., when the model is exact and there are no disturbances.

The stabilizing properties of the control law defined by implementation of Algorithm 2 are established below.

Theorem 4: Let \mathbb{F} represent the set of states for which there exists a control sequence that satisfies (7), (9), and (13). The suboptimal dual-mode MPC law is asymptotically stabilizing with a region of attraction \mathbb{F} .

The proof for this theorem follows closely the proof for Theorem 2; the fact that $L(x, h_L(x)) = 0$ for all $x \in W$ ensures that $\phi(x_{k+1}, \pi_{k+1}) - \phi(x_k, \pi_k) \leq -\mu L(x_k, u_k)$ along trajectories of the controlled system if $x_k \notin W$. Convergence to W in finite time

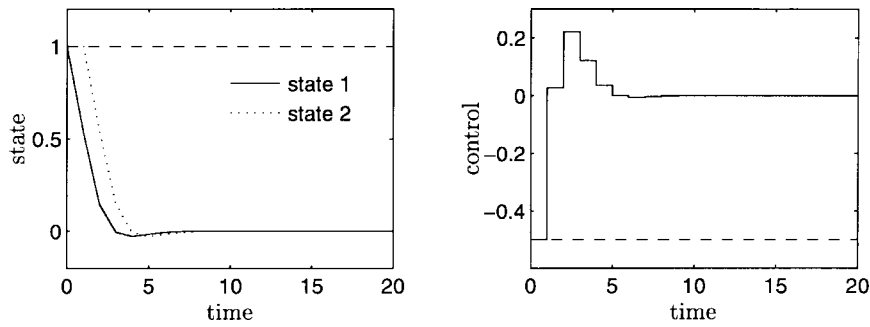


Fig. 1. Example 1: optimal classical MPC.

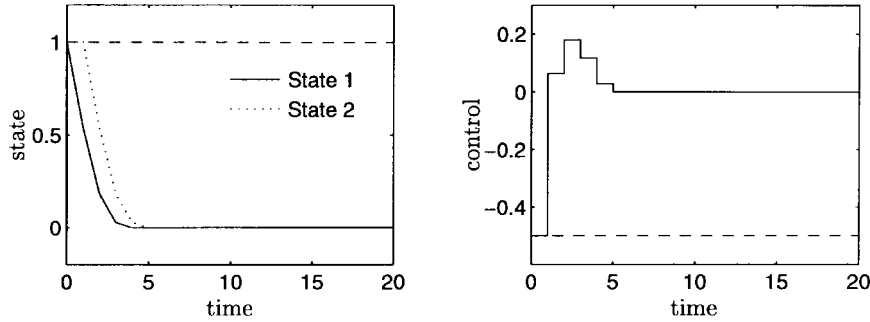


Fig. 2. Example 1: suboptimal classical MPC.

follows, because $\mu > 0$ and $L(\cdot)$ is bounded below by a K function (Assumption A3).

Global or even local minimization of the objective is not required and all the constraints that must be satisfied are inequalities. The computational complexity of the control calculation is substantially reduced.

The suboptimal dual-mode MPC law drives the system state to W in finite time in the presence of model inaccuracies and disturbances as long as the algorithm does not fail. If the local control law $u = h_L(x)$ is robust in the sense that it keeps trajectories that commence in W in W , then the above stability guarantee for dual-mode MPC holds in the presence of model inaccuracies and disturbances provided a feasible solution to the inequalities (7), (9) and (13) can be found. Failure of the algorithm may be caused by large model inaccuracies or disturbances.

V. ILLUSTRATIVE EXAMPLES

We present two simulated examples to illustrate the main ideas of the paper. In the first example, we consider a linear process. In this case, we can easily compute the optimal and suboptimal MPC laws and we compare optimal and suboptimal MPC formulations, in nominal simulations. In the second example, we illustrate suboptimal MPC of a nonlinear process and observe the effects of disturbances and model inaccuracies.

Example 1: The process we consider is linear, unstable, and is described by

$$x_{k+1} = Ax_k + Bu_k \quad (17)$$

where

$$A = \begin{bmatrix} 2 & -0.96 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_k. \quad (18)$$

The stage cost is $L(x, u) = x'x + u'u$. The horizon is $N = 5$, and the constraint sets for the controls and states are $\mathcal{U} = \{u \in \mathbb{R} : |u| \leq 0.5\}$, and $\mathcal{X} = \{x \in \mathbb{R}^n : \|x\|_\infty \leq 1\}$. The initial state

is $x_0 = [1 \ 1]'$, which belongs to the region of attraction for both classical and dual-mode MPC with $N = 5$ and the constraints we use.

For linear systems, the control optimization required for optimal MPC may be cast as a quadratic program. The benefits of suboptimal MPC are thus not as great as with nonlinear systems. Fig. 1 shows the results obtained for optimal classical MPC; the results obtained for the suboptimal version are shown in Fig. 2. This suboptimal version requires solution of a finite set of linear inequalities at each sampling instant (if there are disturbances or model error); a linear program may be employed.

Both control schemes stabilize the process, satisfy the input and state constraints at all times, and have similar performance; performance of the suboptimal scheme depends, of course, on the initial profile.

For the suboptimal version of fixed-horizon dual-mode MPC we set $W = \{x \in \mathbb{R}^n : \|x\|_\infty \leq 0.225\}$, with $h_L(x)$ defined as the unconstrained LQR law $h_L(x) = -Kx$ where $K = [1.4216, -0.795]$. Then $x_k \in W \Rightarrow x_{k+1} = f(x_k, h_L(x_k)) \in W$, because $\|A - BK\|_\infty = 1$ so that W is positively invariant, as required. Also, h_L maps W onto \mathcal{U} and $W \subset \mathcal{X}$. The stage cost is $L(x, u) = \theta(x)[x'x + u'u]$, with $\theta(x)$ defined as in (16). The results obtained with the optimal and suboptimal formulations are presented in Figs. 3 and 4, respectively.

Both control schemes stabilize the process and satisfy the input and state constraints. The trajectories differ considerably; although $x_3 \in W$ in both versions, $\|x_3\|_2$ is considerably less for the suboptimal version than the optimal version which does not cost x in W . Consequently, $\|u_3\|$ is larger and the response less smooth in the optimal version.

Example 2: We consider a single-state nonlinear process, modeled as

$$x_{k+1} = x_k^2 + u_k^3. \quad (19)$$

We implement the suboptimal dual-mode MPC law with horizon $N = 2$, $\mathcal{U} = \{u \in \mathbb{R} : |u| \leq 2\}$, and $\mathcal{X} = \{x \in \mathbb{R}^n : |x| \leq 2.5\}$. The locally stabilizing control law is $h_L(x) = 0$ and $W = \{x \in$

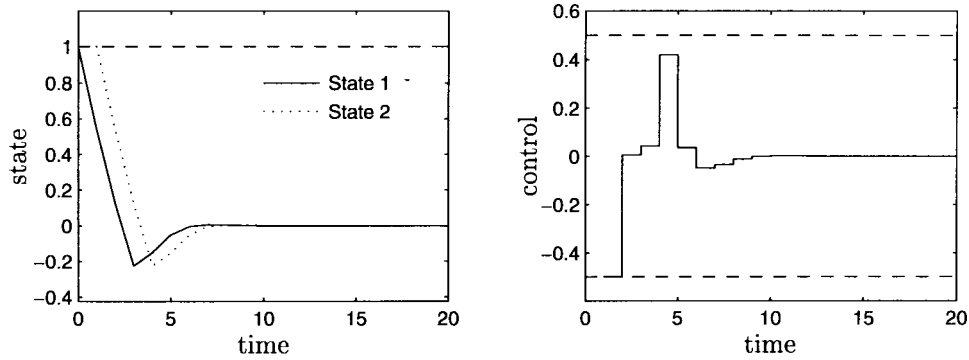


Fig. 3. Example 1: optimal dual-mode MPC.

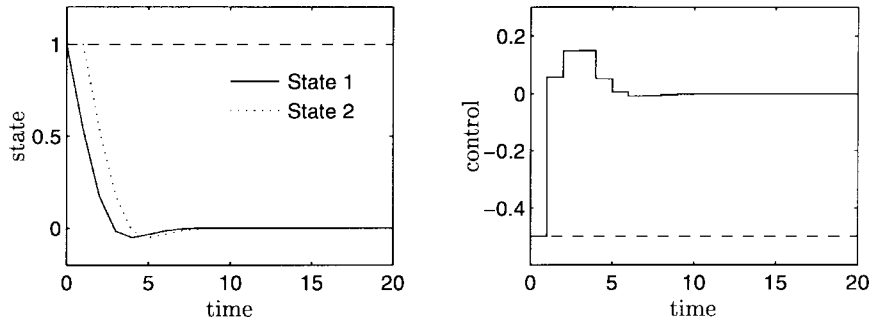


Fig. 4. Example 1: suboptimal dual-mode MPC.

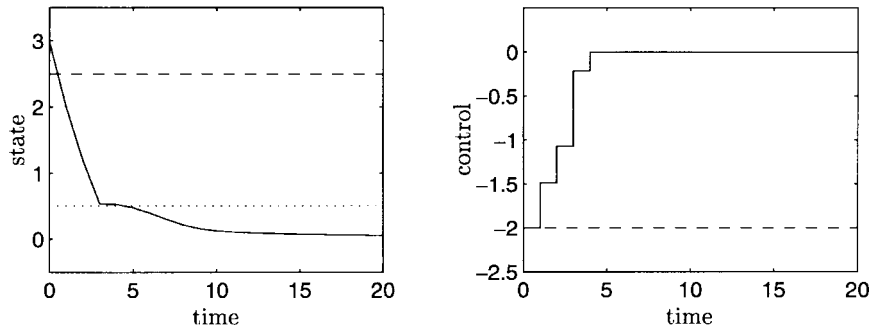


Fig. 5. Example 2: effects of state disturbance MPC.

$\mathbb{R}^n: |x| \leq 0.5$ }. The stage cost is $L(x, u) = \theta(x)[x^2 + u^2]$, with $\theta(x)$ defined as in (16) and we set $\mu = 0.1$. The initial state is $x_0 = 3$, which violates the state constraint, but belongs to the region of attraction for the dual-mode MPC law with the settings we use.

First, we present a simulation, in which a disturbance p_k is added to the system state so that the process is described by

$$x_{k+1} = x_k^2 + u_k^3 + p_{k+1} \quad (20)$$

where $p_k = 1/k$. The local control law $u = h_L(x) = 0$ is robust to disturbances of magnitude no greater than 0.25 in the sense that it keeps trajectories of (20) that commence in W despite such disturbances.

The simulation results are presented in Fig. 5. At time zero, a control sequence π_0 , which merely satisfies the constraints, is calculated. At times one and two, new control sequences π_1 and π_2 need to be computed as π increases the cost and does not satisfy the constraints due to the disturbance. No recalculation is required after time two. At time five, the state enters W and the disturbance has decayed enough to be handled by the robust local control law

$u = h_L(x)$. The local control law then steers the state asymptotically to the origin.

In the second simulation, we illustrate the effects of model inaccuracies and large disturbances. We still use the model of (19), but the process is now described by

$$x_{k+1} = x_k^3 + 0.5u_k^3 + p_{k+1}, \quad k \geq 0 \quad (21)$$

where $p_k = 0$ for all $k \neq 10$, $p_{10} = 1$. The local control law $u = h_L(x)$ is robust to the model inaccuracy and to disturbances of magnitude no greater than 0.375 in the sense that it keeps trajectories of (21), that commence in W despite disturbances of this size.

The simulation results are presented in Fig. 6. At time zero, a control sequence π_0 , is calculated, which merely satisfies the constraints. At time one, a new control sequence π_1 must be computed as π does not satisfy the stability constraints due to the model inaccuracy. No recalculation is required after time one. The state enters W at time three at which time control switches to the local law $u = h_L(x)$ and the stage cost becomes zero. At time ten, a disturbance suddenly hits the process making it impossible to maintain zero cost. The algorithm then fails and we lose the stability

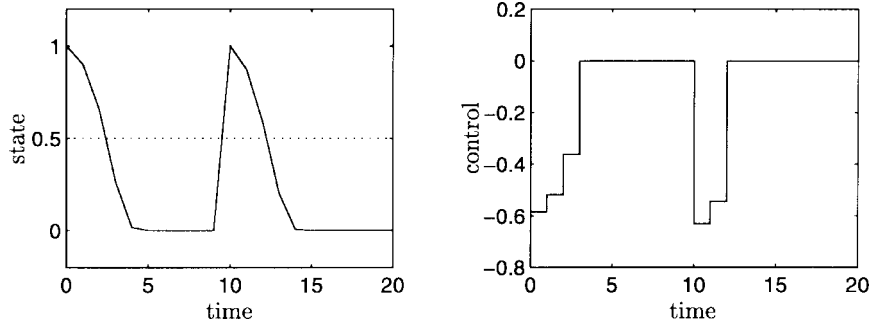


Fig. 6. Example 2: effects of model inaccuracies and state disturbances.

guarantee. We continue control by temporarily abandoning “cost reduction” attempts. At time ten, therefore, a new control sequence π_{10} is calculated, which satisfies the constraints, but yields an increased cost. Although $x_{10} = x_0$, the controller produces $\pi_{10} \neq \pi_0$. No recalculation of the control is required after time ten. The state re-enters W at time 13 and is subsequently steered asymptotically to the origin by the robust local control law $u = h_L(x)$. Although the stability guarantee is lost momentarily at time ten, it is regained immediately afterwards.

VI. CONCLUSIONS

In this paper, we consider optimization problems that arise with the practical implementation of MPC for nonlinear discrete-time systems and show that feasibility rather than optimality suffices for stability. We illustrate this result by establishing stability for suboptimal versions of classical and dual mode MPC; the suboptimal versions have *relatively* modest computational demands. The discontinuous stage cost (15) causes computational difficulties if optimal MPC is employed [9]. This difficulty is avoided in the suboptimal MPC strategies presented above. It is possible to avoid the use of discontinuous stage cost by adding a suitable terminal cost; this will be discussed elsewhere.

APPENDIX

Here, we discuss briefly determination of the set W . If the system is linear and the sets \mathbb{U} and \mathbb{X} are specified by sets of linear inequalities, then W can be chosen to be the output admissible set defined in [8] where a method for constructing this set is given. This is the optimal (largest) set which satisfies the requirements for W .

Suppose then, that $f(\cdot)$ is nonlinear and that the partial derivative $f_z(\cdot) = [f_x(\cdot), f_u(\cdot)]$ with $z := (x, u)$ is Lipschitz continuous with Lipschitz constant c in a neighborhood $C_z = \{z \mid \|z\| \leq \varepsilon\}$ of the origin; this constant is often available. Choose the local stabilizing controller $u = h_L(x)$ to be linear, i.e., $h_L(x) = Kx$ where $A + BK$ is stable where $A := f_x(0, 0)$ and $B := f_u(0, 0)$. Let $g: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be defined by $g(x) := f(x, Kx)$. Then $c(1 + \|K\|)$ is a Lipschitz constant for $g_x(\cdot)$ in $C_x := \{x \mid (x, Kx) \in C_z\}$ so that

$$g(x) = Fx + e(x)$$

where $F := A + BK$ and $\|e(x)\| \leq c_1\|x\|^2$ for all $x \in C_x$ where $c_1 = c(1 + \|K\|)^2$. Choose $Q > 0$ (a suitable choice is obtained if K is designed using linear quadratic optimal control) and $P > 0$ satisfying the Lyapunov equation

$$P = [A + BK]^T P [A + BK] + Q.$$

Let the Lyapunov function $V(\cdot): \mathbb{R}^n \rightarrow \mathbb{R}$ be defined by

$$V(x) = (1/2)x^T Px.$$

Then, for all $x \in C_x$

$$\begin{aligned} V(g(x)) - V(x) &= V(Fx + e(x)) - V(x) = V(Fx) - V(x) + e_1(x) \end{aligned}$$

where $\|e_1(x)\| \leq d\|x\|^3$; $d > 0$ is easily obtained from c_1 and P . Hence,

$$V(g(x)) - V(x) \leq -(1/2)x^T Qx + e_1(x).$$

It is then a simple matter to determine an $\alpha > 0$ such that

$$V(g(x)) - V(x) \leq -(1/4)x^T Qx$$

for all $x \in W_\alpha$ where

$$W_\alpha := \{x \mid x^T Px \leq \alpha\}.$$

This value of α is then reduced, if necessary, to ensure satisfaction of

$$W_\alpha \subset \mathbb{X} \cap C_x, \quad KW_\alpha \subset \mathbb{U}. \tag{22}$$

The resultant W_α satisfies all our requirements for W ; it is positively invariant for the system $x_{k+1} = f(x_k, h_L(x_k))$ and any motion commencing at an initial state in W satisfies the state and control constraints and converges to the origin.

If a Lipschitz constant c_1 for $g_x(\cdot)$ is not known, any α satisfying

$$\max_{x \in W_\alpha} \{V(g(x)) - V(x) + (1/4)x^T Qx\} \leq 0$$

and (22) is acceptable. The maximization is global, but the problem is highly structured and it is known that α sufficiently small satisfies the inequality.

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Robust Adaptive Control of Robots with Arbitrary Transient Performance and Disturbance Attenuation

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Abstract—We consider the tracking problem for robot manipulators with unknown and (possibly) time-varying parameters, which are subject to bounded disturbances. We provide a state feedback adaptive control algorithm which guarantees arbitrary transient performance as well as arbitrary disturbance attenuation. If the disturbances vanish and the parameters remain constant, the proposed controller achieves asymptotic tracking.

Index Terms—Disturbance attenuation, robot manipulators, robust adaptive control, time-varying parameters.

I. INTRODUCTION

We consider the problem of high-performance tracking control of robotics manipulators when their parameters are unknown (and/or time-varying) and they are affected by time-varying disturbances. Our objective is to achieve a state feedback control which guarantees arbitrary transient performance and disturbance attenuation (as long as the actuator power is sufficient) while ensuring zero output tracking error when parameters are constant and disturbances are zero.

The latter property is not provided by robust or H_∞ control techniques such as those proposed in [1]–[4] while it is guaranteed by adaptive controls such as those presented in [5] and [6] (see also [7] for a survey). On the other hand, adaptive controllers may not work properly in presence of time-varying disturbances and

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parameters (see [8]). For these reasons, in [9] a robust adaptive controller was developed which may tolerate time-varying parameters and disturbances, and guarantees asymptotic tracking in presence of constant parameters and vanishing disturbances. However, the transient performance are related to disturbance bounds and speed of parameter variations and cannot be arbitrarily improved.

In this note, we present a robust adaptive state feedback tracking controller which guarantees arbitrary transient performance as well as arbitrary attenuation on the output tracking error of the effects of bounded disturbances and time-varying parameters (both in L_2 and L_∞ sense). Moreover, the proposed controller provides asymptotic output tracking when disturbances vanish and parameters become constant. The control algorithm is obtained suitably modifying the techniques developed in [10] and [11] with reference to single-input single-output systems.

II. MAIN RESULT

Consider the dynamic equations of a rigid manipulator, with $n + 1$ links interconnected by n joints, with possibly time-varying parameters (see [9])

$$B(q, \alpha(t))\ddot{q} + C(q, \dot{q}, \alpha(t))\dot{q} + h(q, \alpha(t)) + f(q, \dot{q}, \alpha(t)) + \frac{\partial B(q, \alpha(t))}{\partial t} \dot{q} = u + d(t) \quad (1)$$

in which the vector $q = [q_1, \dots, q_n]^T$ represents the joint relative displacements, the vector $\alpha(t)$ depends on the kinematic and dynamic parameters of the robot and actuators and enters linearly in the robot equations, the vector u denotes generalized forces (forces or torque) applied at the joints, $B(q, \alpha)$ is the symmetric positive definite inertia matrix, $C(q, \dot{q}, \alpha)\dot{q}$ represents Coriolis and centripetal forces, $(\partial B/\partial t)\dot{q}$ takes into account the time-varying nature of parameters α , and $f(q, \dot{q}, \alpha)$ denotes frictional forces. The disturbance forces are grouped into the vector $d(t)$ which is assumed to be bounded. The vector α is assumed to belong to a known compact set, which for the sake of simplicity is supposed to be a closed ball centered at α_N (the nominal value of α) with radius θ_M (the largest deviation from the nominal value); moreover, it is assumed to be differentiable with bounded time derivative. The deviations of the parameters from the nominal values are indicated by the vector $\theta = \alpha - \alpha_N$. The choice of $C(q, \dot{q}, \alpha(t))$ is not unique; we choose the elements of C as follows [5], [12]:

$$C_{ij} = \frac{1}{2} \left[\dot{q}^T \frac{\partial B_{ij}}{\partial q} + \sum_{k=1}^n \left(\frac{\partial B_{ik}}{\partial q_j} - \frac{\partial B_{jk}}{\partial q_i} \right) \dot{q}_k \right] \quad (2)$$

where $i, j = 1, \dots, n$, so that if α is constant, the matrix

$$\frac{dB(q, \alpha)}{dt} - 2C(q, \dot{q}, \alpha) = \sum_{i=1}^n \frac{\partial B(q, \alpha)}{\partial q_i} \dot{q}_i - 2C(q, \dot{q}, \alpha)$$

is skew-symmetric [5], [12]. As noted in [9], if $\alpha(t)$ is time-varying, with the choice (2) the matrix $\sum_{i=1}^n (\partial B(q, \alpha(t)) / (\partial q_i)) \dot{q}_i - 2C(q, \dot{q}, \alpha(t))$ is still skew-symmetric while the matrix

$$\begin{aligned} & \frac{dB(q, \alpha(t))}{dt} - 2C(q, \dot{q}, \alpha(t)) \\ &= \sum_{i=1}^n \frac{\partial B(q, \alpha(t))}{\partial q_i} \dot{q}_i + \frac{\partial B(q, \alpha(t))}{\partial t} \\ & \quad - 2C(q, \dot{q}, \alpha(t)) \end{aligned} \quad (3)$$

is no longer skew-symmetric. For the robot (1), we define the following problem.