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# On Mappable Nonlinearities in Robustness Analysis

B. R. Barmish and R. Tempo

***Abstract*— When carrying out robustness analysis in the frequency domain, the following fundamental problem arises: Given a description of the uncertain quantities entering the system, at each frequency $\omega$, we need to carry out a mapping into the complex plane. For the special case of multilinear uncertainty structures, the Mapping Theorem greatly facilitates this process and leads to the convex hull of the value set of interest. In this paper, we generalize the class of nonlinear uncertainty structures for which the convex hull can be generated—the so-called Generalized Mapping Theorem goes considerably beyond the multilinear setting. For example, this new framework leads to mappability for large classes of polynomic and nonlinear uncertainty structures. The formulas associated with convex hull generation are seen to be easily implemented in two-dimensional graphics.**

## I. Introduction

This paper concentrates on the mapping of uncertainty into the complex plane. To illustrate, if one begins with a feedback system whose description involves uncertain parameters lying within given bounds, at each frequency $\omega$, it is of interest to find the value set for the corresponding closed-loop polynomial. Having the ability to generate this set in graphics makes it possible to solve robust stability problems in a computer-aided context.

To be more specific, throughout this paper, the framework for value set generation is described as follows: The notation $q$ is used to denote a vector of real uncertain parameters, and $p(s, q)$ is assumed to be a polynomial having coefficients depending continuously on $q$. The given data also consists of lower and upper bounds for each component $q_i$ of $q$. Hence, we use the notation $Q$ for the box describing the domain of $q$. Although $p(s, q)$ is assumed to be a polynomial, it should be noted that the analysis to follow also applies if $p(s, q)$ is a general nonlinear function satisfying the mappability conditions given in Section II. For example, it is of interest to work with rational functions rather than polynomials in a frequency response context.

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The problem of value set generation is now easily described. Namely, given a fixed frequency $\omega \geq 0$, it is of interest to generate the value set

$$p(j\omega, Q) = \{p(j\omega, q) : q \in Q\}.$$

Note that $p(j\omega, Q)$ is the image of the multidimensional box $Q$ under the mapping $p(j\omega, \cdot)$ and is a subset of the complex plane.

A description of $p(j\omega, Q)$ proves to be quite useful for robust stability testing. That is, under mild conditions, robust stability is guaranteed if and only if the zero exclusion condition

$$0 \notin p(j\omega, Q)$$

is satisfied for all $\omega \geq 0$. This type of stability condition goes back quite far in the literature; for example, see [1] for historical purposes and [2] for a more recent exposition.

For the case when the coefficients of $p(s, q)$ are affine linear functions, the robust stability problem is said to have a "polytopic structure" and the value set $p(j\omega, Q)$ is easily shown to be a convex polygon; e.g., see [2]. However, for the more general case when $q$ enters nonlinearly, only very limited results are available. In this regard, perhaps the strongest result to date is the Mapping Theorem: For the case of multilinear dependence on the uncertain parameters, this theorem provides a simple recipe for generation of the convex hull of the value set. In other words, the Mapping Theorem makes it possible to obtain the tightest possible convex bound which we denote by $\operatorname{conv} p(j\omega, Q)$. In fact, this convex hull turns out to be a simple polygon whose vertices are obtained from the set of vertices of the bounding box $Q$. More precisely, if we denote the set of vertices of $Q$ by $\{q^1, q^2, \cdots, q^L\}$, then the Mapping Theorem (see [3] for further details) tells us that

$$\operatorname{conv} p(j\omega, Q) = \operatorname{conv}\{p(j\omega, q^1), p(j\omega, q^2) \cdots, p(j\omega, q^L)\}.$$

Given the background described above, the motivation for this paper is derived from the following question: Is there a generalization of the Mapping Theorem which accommodates much larger classes of uncertainty structures? For example, can we handle rich classes of polynomials or more general nonlinear dependencies on uncertain parameters? The point of view taken in this paper is that this question is ill-posed unless computability considerations are taken into account. Without such considerations, in principle, it is possible to allow arbitrary nonlinear dependencies because gridding the box $Q$ may be a viable option. Therefore, to rule out computationally intractable options, one needs to declare *a priori,* either explicitly or implicitly, what type of computations are acceptable. For example, use of the Mapping Theorem involves the implicit assumption that one is willing to evaluate $p(j\omega, q)$ at each vertex $q^i$ of $Q$.

In Section II, the issues involving computational tractability are precisely formulated. At levels of computational effort comparable with those associated with the Mapping Theorem, we see that convex hull generation is tractable for much larger classes of nonlinearities than heretofore considered. The main result of this paper, Theorem 3.1, is paraphrased as follows: When an uncertain polynomial $p(s, q)$ satisfies a certain "mappability" condition, we can construct a distinguished uncertainty function $q^*(\omega, \theta)$, parameterized by a single parameter $\theta \in [0, 2\pi]$. Moreover, the convex hull of the value set $p(j\omega, Q)$ is the convex hull of the "boundary trace" obtained by sweeping the angle $\theta$ from 0 to $2\pi$ while plotting $p[j\omega, q^*(\omega, \theta)]$ in the complex plane. That is, the convex hull of this plot is precisely the convex hull of $p(j\omega, Q)$. In the sequel, we refer to this result as the Generalized Mapping Theorem and argue that the desired boundary trace is easy to generate in graphics.

Finally, it is important to explain how the result in this paper reduces to the Mapping Theorem for the special case of multilinear mappings. In this situation, it turns out that for every $\theta \in [0, 2\pi]$, the distinguished uncertainty function $q^*(\omega, \theta)$ coincides with one of the vertices $q^i$ of $Q$. Hence, consistent with existing theory, the convex hull of the value set is a polygon. However, for more general uncertainty structures, the convex hull of the boundary trace is typically not polygonal.

## II. The Class $\mathcal{F}$ of Mappable Polynomials

Since value set generation involves working with one frequency at a time, in the sequel, we consider $\omega \geq 0$ as fixed. Now, for the uncertain polynomial $p(s, q)$, we need to declare what operations are *computationally acceptable*. We begin with a heuristic explanation: Roughly speaking, we view an operation as "elementary" if the user is willing to execute it parametrically in the boundary trace parameter $\theta$. For example, most users recognize the fact that up to rather large data dimension, a linear program is rapidly executable with modest computing power. Hence, for most modern machines, it would be reasonable to consider a value set generation method involving the solution of a linear program (of predeclared size) for each $\theta \in [0, 2\pi]$. Of course, analogous to frequency sweep methods, the understanding is that a computer implementation would only involve a finite sweep for $\theta$.

To make the ideas above more precise, we take $\mathcal{F}$ to be a user-defined class of functions on $Q$ for which maximization is deemed elementary. To illustrate, consider the scenario associated with the Mapping Theorem. If the evaluation of multilinear functions at the extreme points of $Q$ is deemed elementary, then, for a given dimension of $q$, the class $\mathcal{F}$ would include all functions $f : Q \rightarrow \mathbf{C}$ which are multilinear. In other words, multilinear function maximization is an operation which is deemed reasonable to perform parametrically with respect to $\theta$. It is important to emphasize the point of view being taken—the class $\mathcal{F}$ is determined by the available computing power and the problem dimension. On one end of the spectrum, $\mathcal{F}$ can include rather arbitrary nonlinear functions when the dimension of $q$ is small; this follows from the fact that gridding of the $Q$ box is a viable option for maximization with only a few variables. As the number of variables increases, however, gridding becomes infeasible and $\mathcal{F}$ only includes functions for which a global maximum can be guaranteed.

We are now prepared to define the class of uncertain polynomials to which the Generalized Mapping Theorem applies. Immediately following the definition, we argue that there are large classes of "mappable" nonlinearities which are not covered by the Mapping Theorem and require levels of computational efforts which are comparable to that associated with the multilinear case.

### A. Mappable Polynomials

For pairs of complex numbers $z_1, z_2 \in \mathbf{C}$, we use the inner product notation

$$\langle z_1, z_2 \rangle \doteq \operatorname{Re} z_1 \operatorname{Re} z_2 + \operatorname{Im} z_1 \operatorname{Im} z_2.$$

Then, the uncertain polynomial $p(s, q)$ is said to be mappable if the following condition is satisfied: Given any fixed frequency $\omega \geq 0$ and any complex number $z \in \mathbf{C}$, the function

$$f_z(\omega, q) \doteq \langle z, p(j\omega, q) \rangle$$

is in $\mathcal{F}$. That is, at each frequency $\omega$, the generation of

$$q^*(\omega) \in \arg \max_{q \in Q} f_z(\omega, q)$$

is deemed elementary, and we obtain

$$f_z[\omega, q^*(\omega)] \doteq \max_{q \in Q} f_z(\omega, q).$$

### B. Richness of the Mappable Class

For the remainder of this section, the main objective is to provide examples of rich classes of uncertain polynomials with nonlinear dependence on $q$ which violate the preconditions of the Mapping Theorem but are nevertheless mappable for typical $\mathcal{F}$.

### C. Example (Separable Nonlinearities)

To describe a simple class of nonlinear functions which would typically be included in $\mathcal{F}$, suppose that the function $f_z(\omega, q)$ can be decomposed into a sum

$$f_z(\omega, q) = f_{z,1}(\omega, q_1) + f_{z,2}(\omega, q_2) + \cdots + f_{z,\ell}(\omega, q_\ell)$$

with each $f_{z,i}$ depending on one distinct component $q_i$ of $q$. In this case, it is easy to see that

$$\max_{q \in Q} f_z(\omega, q) = \max_{q_1} f_{z,1}(\omega, q_1) + \max_{q_2} f_{z,2}(\omega, q_2) + \cdots + \max_{q_\ell} f_{z,\ell}(\omega, q_\ell).$$

Hence, the maximization of $f_z(\omega, q)$ amounts to $\ell$ independent line searches and motivates the inclusion of such "separable" functions in $\mathcal{F}$.

To illustrate how this situation arises in a systems context, we consider an uncertain complex function of the form

$$p(s, q) = p_1(s)\psi_1(s, q_1) + p_2(s)\psi_2(s, q_2) + \cdots + p_\ell(s)\psi_\ell(s, q_\ell)$$

where each $p_i(s)$ is a fixed polynomial, the uncertain parameter vector $q$ is known to lie within a given box $Q$, and each $\psi_i(s, q_i)$ is a general nonlinearity; for example, $\psi_1(s, q_1)$ can be a saturation function and $\psi_2(s, q_2)$ can represent a delay.

To justify mappability of $p(s, q)$, we fix $z \in \mathbf{C}$ and form the function $f_z(\omega, q)$. Indeed, via a straightforward calculation, it is easy to verify that the function $f_z(\omega, q)$ is decomposable as

$$f_z(\omega, q) = \sum_{i=1}^{\ell} f_{z,i}(\omega, q)$$

where

$$f_{z,i}(\omega, q) \doteq \langle z, p_i(j\omega)\psi_i(j\omega, q_i) \rangle.$$

Moreover, since each of the functions $f_{z,i}(\omega, q)$ depends on only one component $q_i$ of $q$, we conclude that $p(s, q)$ is mappable whenever the class $\mathcal{F}$ includes functions which can be maximized via $\ell$ independent line searches.

### D. Example (Both Separable Nonlinearities and Multilinearities)

The class of uncertainty structures in the example above can be generalized to include multilinear effects. To this end, we partition $q$ as $q = (u, v)$ and consider a function of the form

$$f_z(\omega, u, v) = \sum_{i=1}^{m} g_{z,i}(\omega, u) h_{z,i}(\omega, v_i)$$

with each function $g_{z,i}(\omega, u)$ being multilinear in $u$ and $h_{z,i}(\omega, v_i)$ being a nonlinear function depending only on $v_i$. Observe that we recover the previous example by setting all of the $g_{z,i}(\omega, u) \equiv 1$ and by having the Mapping Theorem situation (pure multilinearity) by setting all of the $h_{z,i}(\omega, v_i) \equiv 1$. Letting $\{u^k\}$ denote the set of vertices associated with the bounds for $u$, the following set of equalities motivates inclusion of this class of functions in $\mathcal{F}$:

$$\begin{aligned}\max_{u,v} f_z(\omega, u, v) &= \max_{v} \max_{u} f_z(\omega, u, v)\\ &= \max_{v} \max_{k} f_z(\omega, u^k, v)\\ &= \max_{k} \max_{v} f_z(\omega, u^k, v).\end{aligned}$$

This formula says that we obtain a separate optimization problem for each vertex $u^k$. Moreover, the nature of the optimization is exactly as in the preceding example—a simple line search for each component $v_i$ of $v$. In other words, for fixed $k$, if $v$ has $m$ components, we have

$$\max_{v} f_z(\omega, u^k, v) = \sum_{i=1}^{m} g_{z,i}(\omega, u^k) \max_{v_i} h_{z,i}(\omega, v_i).$$

To complete the discussion for this class of functions, we make the identification between the optimization problem above and satisfaction of the mappability condition for uncertain quasipolynomials which have both multilinear parameter dependence and delays. Indeed, we consider

$$\begin{aligned}p(s, u, v) =& p_1(s, u)e^{-v_1 s} + p_2(s, u)e^{-v_2 s}\\ &+ \cdots + p_m(s, u)e^{-v_m s}\end{aligned}$$

where each $p_i(s, u)$ has coefficients depending multilinearly on $u$, and $v$ is the uncertain delay vector. Now, for any fixed complex number $z$, it is a straightforward calculation to show that the function $f_z(\omega, u, v)$ is decomposable as

$$f_z(\omega, u, v) = \sum_{i=1}^{m} g_{z,i}(\omega, u) h_{z,i}(\omega, v_i)$$

with each $g_{z,i}$ being multilinear and each $h_{z,i}$ depending only on $v_i$.

### E. A More General Class

The analysis given for the class of functions above extends to a more general class $\mathcal{F}$. Partitioning $q$ again as $q = (u, v)$, we begin with a function of the form

$$f_z(\omega, u, v) = \sum_{i=1}^{m} f_{z,i}(\omega, u, v_i)$$

with each of the functions $f_{z,i}(\omega, u, v_i)$ depending on the single component $v_i$ of $v$ and being multilinear in $u$ for fixed $v_i$. As in the preceding example, it is easy to see that the maximum of $f_z(\omega, u, v)$ can be obtained by working with one extreme point $u^k$ at a time. Then, for fixed $u = u^k$, we obtain

$$\max_{v} f_z(\omega, u^k, v) = \sum_{i=1}^{m} \max_{v_i} f_{z,i}(\omega, u^k, v_i).$$

Consequently, we introduce the complex number $z$ and generate $f_z(\omega, u, v)$ as in the preceding example. Since $q = (u, v)$ has dimension $\ell$, there are $2^{\ell - m}$ extreme points for $u$ with each extreme requiring $m$ line searches. Therefore, the total quantity $M = m2^{\ell - m}$ is relevant to the decision whether to include such functions in $\mathcal{F}$.

### F. Example

To provide a concrete illustration of the situation above, we consider the uncertain polynomial

$$p(s, q) = s^3 + a_2(q)s^2 + a_1(q)s + a_0(q)$$

with

$$\begin{aligned}a_0(q) &= \varphi_1(q_3) + \varphi_2(q_3)q_1 + q_1 q_2 \varphi_3(q_4) + 5\\ a_1(q) &= 2\varphi_2(q_3)q_2 + q_1 q_2 \varphi_4(q_4) + 20\\ a_2(q) &= 4q_3 + q_2 + 2q_1 q_2 + 0.5\end{aligned}$$

and each of the $\varphi_i$ being any nonlinear function; in fact, these functions can even be described in tabular form.

To make appropriate identification between $p(s, q)$ and the class of functions associated with $f_z(\omega, u, v)$ above, we take $(u_1, u_2) = (q_1, q_2)$ and $(v_1, v_2) = (q_3, q_4)$. Then, for fixed frequency $\omega \geq 0$ and

$$f_z(\omega, u, v) = \langle z, p(j\omega, u, v)\rangle$$

a straightforward calculation leads to the decomposition

$$f_z(\omega, u, v) = f_{z,1}(\omega, u, v_1) + f_{z,2}(\omega, u, v_2)$$

where

$$\begin{aligned}f_{z,1}(\omega, u, v_1) &\doteq \langle z, p_1(j\omega, u, v_1)\rangle\\ f_{z,2}(\omega, u, v_2) &\doteq \langle z, p_2(j\omega, u, v_2)\rangle\end{aligned}$$

and

$$\begin{aligned}p_1(s, u, v_1) \doteq& \varphi_1(v_1) + \varphi_2(v_1)u_1\\ &+ 2\varphi_2(v_1)u_2 s + (u_2 + 4v_1)s^2\\ &+ s^3 + 0.5s^2 + 20s + 5\\ p_2(s, u, v_2) \doteq& \varphi_3(v_2)u_1 u_2\\ &+ \varphi_4(v_2)u_1 u_2 s + 2u_1 u_2 s^2.\end{aligned}$$

Moreover, the decomposition above satisfies the conditions required for membership in $\mathcal{F}$.

Finally, it should be noted that examples of the sort above can be generalized; e.g., if each $f_{z,i}$ depends on $n_i$ distinct $v_i$, then the user can declare an acceptable gridding dimension $n_{\max}$ and allow $\mathcal{F}$ to include all functions having the property that

$$\max_i n_i \leq n_{\max}.$$

For example, one takes $n_{\max} = 2$ whenever two-dimensional searches are acceptable.

## III. The Generalized Mapping Theorem

To state the main result, we need to describe the generation of the distinguished uncertainty $q^*(\omega, \theta)$. To this end, for fixed frequency $\omega \geq 0$, we assume that $p(s, q)$ is mappable and consider the function $f_z(\omega, q)$ with $z = e^{j\theta}$. That is

$$f_{e^{j\theta}}(\omega, q) \doteq \langle e^{j\theta}, p(j\omega, q)\rangle.$$

Note that mappability of $p(s, q)$ guarantees that for each $\theta \in [0, 2\pi]$, we can generate some distinguished uncertainty function $q^*(\omega, \theta)$

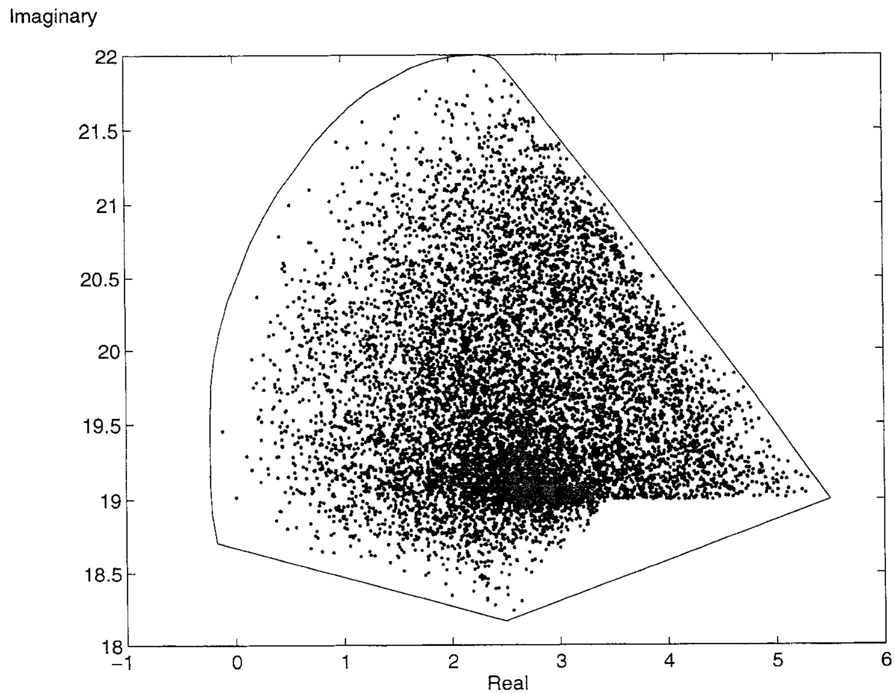


Fig. 1. Application of the theorem to example II-F.

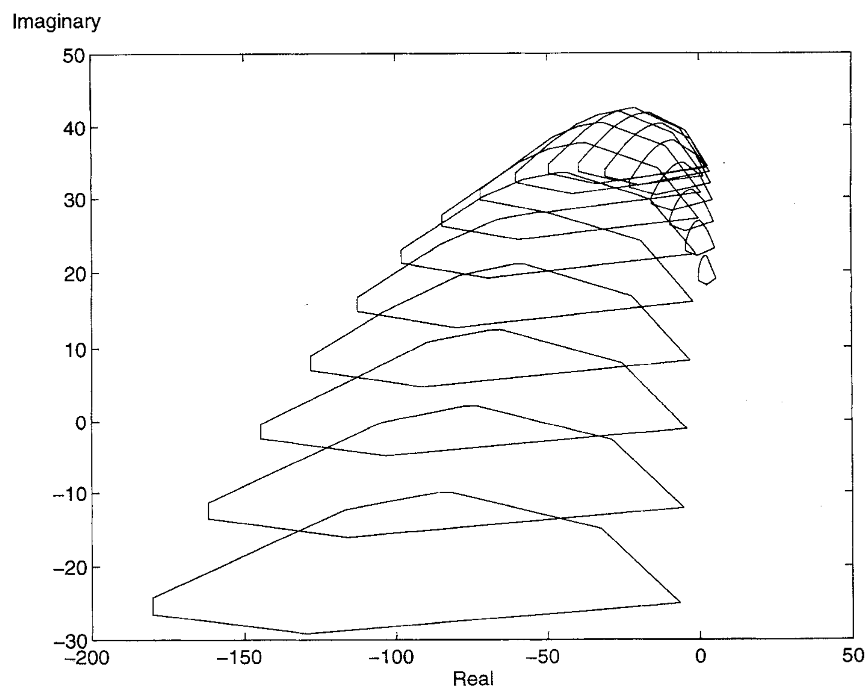


Fig. 2. Robust stability test for example II-F.

achieving the maximum of $f_z(\omega, q)$. Now, we call $p[j\omega, q^*(\omega, \theta)]$ a boundary trace function for $p(s, q)$ at frequency $\omega$. Although this function can be nonunique, the results to follow do not depend on the choice of

$$q^*(\omega, \theta) \in \arg \max_{q \in Q} f_{e^{j\theta}}(\omega, q).$$

Furthermore, in some cases, this function can be discontinuous with range consisting of a finite set of points in the complex plane. For example, when we recover the Mapping Theorem as a special case, this is precisely the situation which occurs.

*Theorem 3.1:* Assume $p(s, q)$ is mappable with distinguished uncertainty $q^*(\omega, \theta)$ and corresponding boundary trace function $p[jw, q^*(\omega, \theta)]$. Then, for each fixed frequency $\omega \geq 0$, it follows that

$$\operatorname{conv} p(j\omega, Q) = \operatorname{conv}\{p[j\omega, q^*(\omega, \theta)] : \theta \in [0, 2\pi]\}.$$

*Proof:* In the proof to follow, it is convenient to use the shorthand notation $\mathbf{P}_\omega$ to denote the right-hand side above. First, note that $\operatorname{conv} p(j\omega, Q)$ is easily seen to be a subset of $\mathbf{P}_\omega$ because $q^*(\omega, \theta) \in Q$ for all $\theta \in [0, 2\pi]$. Therefore, to complete the proof, we fix some $z_0 \in \operatorname{conv} p(j\omega, Q)$ and must show that $z_0 \in \mathbf{P}_\omega$.

Proceeding by contradiction, if $z_0 \notin \mathbf{P}_\omega$, the Separating Hyperplane Theorem (for example, see [4]) guarantees that the point $z_0$ can be strictly separated from the closed convex set $\mathbf{P}_\omega$. Hence, there exists some nonzero complex number $\eta \in \mathbf{C}$ such that

$$\langle \eta, z_0 \rangle > \langle \eta, \mathbf{p} \rangle$$

for all $\mathbf{p} \in \mathbf{P}_\omega$. Now taking

$$\theta_0 \doteq \arg \eta$$

we divide both sides above by $|\eta|$ and obtain

$$\langle e^{j\theta_0}, z_0 \rangle > \langle e^{j\theta_0}, \mathbf{p} \rangle$$

for all $\mathbf{p} \in \mathbf{P}_\omega$. Equivalently

$$\langle e^{j\theta_0}, z_0 \rangle > \max_{\mathbf{p} \in \mathbf{P}_\omega} \langle e^{j\theta_0}, \mathbf{p} \rangle. \tag{1}$$

This is the inequality to be contradicted.

Indeed, using the fact that a linear function on $p(j\omega, Q)$ and $\operatorname{conv} p(j\omega, Q)$ has the same maximum value, we generate the chain of inequalities

$$\begin{aligned}
\langle e^{j\theta_0}, z_0 \rangle &\le \max_{z \in \operatorname{conv} p(j\omega, Q)} \langle e^{j\theta_0}, z \rangle \\
&= \max_{z \in p(j\omega, Q)} \langle e^{j\theta_0}, z \rangle \\
&= \max_{q \in Q} \langle e^{j\theta_0}, p(j\omega, q) \rangle \\
&= \max_{q \in Q} f_{e^{j\theta_0}}(\omega, q) \\
&= f_{e^{j\theta_0}}[\omega, q^*(\omega, \theta_0)] \\
&= \langle e^{j\theta_0}, p[j\omega, q^*(\omega, \theta_0)] \rangle \\
&\le \max_{\mathbf{p} \in \mathbf{P}_\omega} \langle e^{j\theta_0}, \mathbf{p} \rangle.
\end{aligned}$$

The proof is now complete because this inequality contradicts (1).

## IV. Numerical Example

In this section, we illustrate the application of the Generalized Mapping Theorem. To this end, we consider the nonlinear uncertainty structure in the example of Section II-F and assume uncertainty bounds $q_i \in [0, 1]$ for $i = 1, 2, 3, 4$ and specific nonlinearities

$$\begin{aligned}
\varphi_1(q_3) &= q_3^3 + 2q_3^2 \\
\varphi_2(q_3) &= \cos 2q_3 \\
\varphi_3(q_4) &= -(q_4 - 0.5)^2 \\
\varphi_4(q_4) &= \cos q_4.
\end{aligned}$$

In Fig. 1, the convex hull of the value set is shown for frequency $\omega = 1$. For validation purposes, this figure also includes a plot of 10 000 sample points which were obtained via random Matlab evaluations of $p(j\omega, q)$; a uniform distribution over [0, 1] was used for each component $q_i$ of $q$. It is interesting to note that the outward curvature of the boundary of the convex hull is consistent with the fact that the Mapping Theorem cannot be used to obtain the desired convex hull.

It is also important to recall that in a robust stability context, the convex hull can be readily exploited. That is, if $p(s, q)$ has invariant degree and $p(s, q^0)$ is stable for some $q^0 \in Q$, then satisfaction of the zero exclusion condition $0 \notin \operatorname{conv} p(j\omega, Q)$ for all $\omega \ge 0$ guarantees robust stability. For the example at hand, monicity of $p(s, q)$ guarantees invariant degree, and it is easily verified that with nominal uncertainty $q = q^0 = 0$, the polynomial $p(s, q^0)$ is stable. After carrying out a preliminary frequency sweep while plotting $\operatorname{conv} p(j\omega, Q)$, it was determined that for robust stability purposes, the critical range is $1 \le \omega \le 5$. In Fig. 2, the plot is shown with a frequency separation $\Delta\omega = 0.25$. Since zero is excluded from the convex hull at all frequencies, this family of polynomials is deemed to be robustly stable.

## V. Conclusion

In this paper, the notion of mappability was introduced. This demonstrated that one can handle much larger classes of uncertainty structures than those addressed by the Mapping Theorem. This work suggests that an approach based on convexification may be fruitful for even more complicated robustness problems.

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# Modeling of Linear Fading Memory Systems

J. R. Partington and P. M. Mäkilä

***Abstract*—Motivated by questions of approximate modeling and identification, we consider various classes of linear time-varying bounded-input–bounded output (BIBO) stable fading memory systems and prove some characterizations of them. These include fading memory systems, in general, almost periodic systems, and asymptotically periodic systems. We also show that norm and strong convergence coincide for BIBO stable causal fading memory systems.**

## I. Introduction

Recently, an intense research effort has taken place in the emerging field of identification for robust control [21], [9], [20], [6], [17], [24], [7], [15], [14]. Both stochastic and nonstochastic approaches to

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