A UNIVERSAL FIGURE OF MERIT FOR STOCHASTIC FIRST ORDER FILTERS†

Vincent Winstead and B. Ross Barmish

ECE Department, University of Wisconsin, Madison, WI 53706.

ABSTRACT

The focal point of this paper is a new result on the probabilistic robustness of a stochastic first order filter. For a first order filter transfer function, $G(s,\tau)$, we allow a class of probability distributions $\mathcal{F}$ for the time constant τ and consider the following question: Given frequency $\omega \geq 0$ and unknown probability distribution $f \in \mathcal{F}$, to what extent can the expected filter gain $g(\omega,\tau) = |G(j\omega,\tau)|$ deviate from some desired nominal value, $g(\omega,\tau_0)$? It turns out that the deviations of concern are surprisingly low. For example, with 20% variation in τ, the expected filter gain deviates from $g(\omega,\tau_0)$ by no more than 0.4% of the zero frequency gain. In addition to performance bounds such as this, we also provide a so-called *universal figure of merit*. The word "universal" is used because the performance bound attained holds independently of the nominal τ_0, the frequency $\omega \geq 0$ and the admissible probability distributions $f \in \mathcal{F}$.

1. Introduction and Formulation

The problem considered in this paper is motivated by a new line of research involving Monte Carlo analysis of electrical circuits; e.g., see [5], [7] and [8]. In contrast to more classical Monte Carlo approaches to simulation such as in [1] - [4], we assume little a priori information about the probability distribution of the uncertain circuit parameters. In fact, the starting point for this new theory is the same as in robustness analysis — only bounds on the uncertain parameters are assumed a priori. In view of this setup, a certain type of "distributional robustness" is sought; e.g., see [5]. That is, the performance limits which are obtained apply for an entire family of probability distributions $\mathcal{F}$ rather than a single assumed distribution $f \in \mathcal{F}$. It often turns out to be the case that this new approach leads to probabilistic assessments of performance which differ considerably from the ones! obtained in a more classical Monte Carlo setting; for example, see [7] for an illustration in the context of circuits. To motivate the problem under consideration, we consider the simple circuit in Figure 1.0.1 below.

†Funding for this research was provided by the National Science Foundation under Grants ECS-9711590 and ECS-9811051.

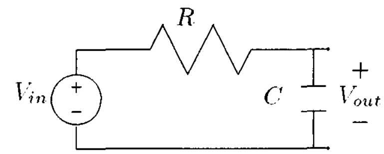

Figure 1.0.1: First Order RC Filter

With the output voltage measured across the capacitor and time constant $\tau = RC$, the filter transfer function is

$$G(s,\tau) = \frac{1}{1+s\tau}$$

and the associated gain at frequency $\omega \in \Omega \doteq [0,\infty)$ is given by

$$g(\omega,\tau) \doteq |G(j\omega)| = \frac{1}{\sqrt{1+\omega^2\tau^2}}.$$

With uncertain circuit parameters R and C, it is important to know how performance will vary when these components are manufactured to some specified tolerance. Within this context, it is of interest to describe the so-called *envelope* of frequency responses. A fundamental issue faced by the circuit designer is the captured by the following question: If there is no statistical description of the uncertain parameters entering a system, is there a way to carry out a Monte Carlo simulation to obtain meaningful predictions? In this context, the issue is what probability distribution to use to generate random circuit parameters. The line of research associated with this paper is aimed at the development of Monte Carlo simulation techniques which can be used when there are bounds but no statistics for the underlying circuit parameters. To this end, this paper formulates and solves an optimization problem whose solution describes the range of "distributionally robust" expected filter gains.

In the sense to be described in Section 2, we obtain a *universal figure of merit* for a class of first order stochastic filters. The word "universal" is used because the performance bound obtained holds independently of the nominal value filter parameter $\tau = \tau_0$, the probability distribution $f \in \mathcal{F}$ for deviations in τ and the frequency $\omega \in \Omega$. For the example of the RC filter in Figure 1.0.1, this universal result indicates that the expected filter gain does not deviate from the low frequency gain by more than about 12%.

2. Formulation

We consider the first order filter with transfer function $G(s,\tau)$ above and uncertain time constant τ. The uncertainty in the time constant τ as described below.

2.1 Admissible Probability Distributions: It is assumed that τ is a random variable expressed as

$$\tau \doteq \tau_0 + \Delta\tau$$

with nominal $\tau_0 \in \mathcal{T}_0 \doteq (0,\infty)$ and uncertainty bound

$$|\Delta\tau| \leq \Delta\tau_{max}.$$

It is assumed that τ has an unknown probability density function $f(\tau)$ which is supported in the interval

$$\mathcal{T} \doteq [\tau_0 - \Delta\tau_{max}, \tau_0 + \Delta\tau_{max}],$$

symmetric about its mean τ_0 and non-increasing in $|\tau - \tau_0|$. We write $f \in \mathcal{F}_\lambda$ to denote an *admissible probability density function* $f(\tau)$. Given any $f \in \mathcal{F}_\lambda$, the resulting random time constant is denoted as τ^f. It is noted that this model of uncertainty is further described in [5] and [8]. Finally, we express the uncertainty bound $\Delta\tau_{max}$ as

$$\Delta\tau_{max} = \lambda\tau_0$$

with

$$0 \leq \lambda \leq 1.$$

In other words, by working with λ, we quote uncertainty $\Delta\tau$ as a percentage of the nominal time constant τ_0.

2.2 Expected Filter Gain Envelopes: For the filter transfer function $G(s,\tau)$ with gain $g(\omega,\tau)$ and probability density function $f \in \mathcal{F}_\lambda$, at frequency $\omega \in \Omega$, we focus attention on the *expected gain*

$$\mathcal{E}g(\omega,\tau^f) \doteq \int_{\tau\in\mathcal{T}} f(\tau)g(\omega,\tau)\, d\tau$$

with its *upper envelope*

$$\mathcal{E}^+g(\omega,\tau^f) \doteq \sup_{f\in\mathcal{F}_\lambda} \mathcal{E}g(\omega,\tau^f)$$

and its *lower envelope*

$$\mathcal{E}^-g(\omega,\tau^f) \doteq \inf_{f\in\mathcal{F}_\lambda} \mathcal{E}g(\omega,\tau^f).$$

2.3 Universal Figure of Merit: For a given level of uncertainty $\lambda \in [0,1]$, we seek to compute the maximum possible deviation between the expected gain and the nominal gain. Since we seek to compute this maximum with respect to all possible $f \in \mathcal{F}_\lambda$, $\tau_0 \in \mathcal{T}_0$, and $\omega \in \Omega$, the resulting figure of merit is said to be universal. That is, we seek to compute the *universal figure of merit*

$$UFOM(\lambda) \doteq \sup_{f\in\mathcal{F}_\lambda,\tau_0\in\mathcal{T}_0,\omega\in\Omega} |\mathcal{E}g(\omega,\tau^f) - g(\omega,\tau_0)|.$$

The next section addresses the solution of the envelope and UFOM problems.

3. Solution of the Envelope Problem

The first step in the solution involves the so-called Truncation Principle; e.g. [5]. Some preliminaries are required.

3.1 Truncated Uniform Distributions: For notational convenience, let

$$T_\lambda \doteq [-\lambda\tau_0, \lambda\tau_0].$$

Then, given any $t \in T_\lambda$, the corresponding *truncated uniform distribution* $u^t(\tau)$ is non-zero with τ in the delay interval $[\tau_0 - t, \tau_0 + t]$ with value

$$u_t(\tau) \equiv \frac{1}{2t}.$$

In other words, $u^t(\tau)$ is uniformly distributed on the subinterval $[\tau_0 - t, \tau_0 + t]$ of T_λ. In this way, we obtain a subclass of probability density functions

$$\mathcal{U}_\lambda^t \doteq \{u^t : t \in T_\lambda\} \subset \mathcal{F}_\lambda$$

Now, in accordance with the so-called Truncation Principle, the optimization of expected gain with respect to $f \in \mathcal{F}_\lambda$ is equivalent to optimization with respect to $t \in T_\lambda$. That is,

$$\begin{aligned}\sup_{f\in\mathcal{F}_\lambda} \mathcal{E}g(\omega,\tau^f) &= \sup_{t\in T_\lambda} \mathcal{E}g(\omega,\tau^t) \\ &= \sup_{t\in T_\lambda} \frac{1}{2t}\int_{\tau_0-t}^{\tau_0+t} \frac{1}{\sqrt{1+\omega^2\tau^2}}d\tau\end{aligned}$$

and

$$\begin{aligned}\inf_{f\in\mathcal{F}_\lambda} \mathcal{E}g(\omega,\tau^f) &= \inf_{t\in T_\lambda} \mathcal{E}g(\omega,\tau^t) \\ &= \inf_{t\in T_\lambda} \frac{1}{2t}\int_{\tau_0-t}^{\tau_0+t} \frac{1}{\sqrt{1+\omega^2\tau^2}}d\tau\end{aligned}$$

where τ^t is the random variable with probability distribution u^t.

3.2 Change of Variables: To simplify notation, we introduce the change of variables, $x \doteq \omega\tau$, $x_0 \doteq \omega\tau_0$, and $\hat{t} \doteq \omega t$. This leads to the problem of optimizing

$$\begin{aligned}\Phi(x_0, \hat{t}) &\doteq \frac{1}{2\hat{t}} \int_{x_0-\hat{t}}^{x_0+\hat{t}} \frac{1}{\sqrt{1+x^2}} dx \\ &= \frac{1}{2\hat{t}} \Big\{ \ln(x_0 + \hat{t} + \sqrt{1+(\hat{t}+x_0)^2}) \\ &\qquad + \ln(-x_0 + \hat{t} + \sqrt{1+(x_0-\hat{t})^2}) \Big\}\end{aligned}$$

subject to $\hat{t} \in \hat{T}_\lambda \doteq [-\lambda\omega x_0, \lambda\omega x_0]$.

3.3 Optimization Plots: Via a series of computations, for each $x_0 > 0$, we first obtain

$$\hat{t}_-(x_0, \lambda) \in \arg\min_{\hat{t} \in \hat{T}_\lambda} \Phi(x_0, \hat{t})$$

and

$$t_+(x_0, \lambda) \in \arg\max_{\hat{t} \in \hat{T}_\lambda} \Phi(x_0, \hat{t}).$$

This leads to the frequency-independent optima $\Phi_\lambda(x_0, \hat{t}_-(x_0, \lambda))$ and $\Phi_\lambda(x_0, \hat{t}_+(x_0, \lambda))$. Subsequently, we obtain the expected gain errors

$$\epsilon^+(x_0, \lambda) = \Phi(x_0, t_+(x_0, \lambda)) - \frac{1}{\sqrt{1+x_0^2}}$$

and

$$\epsilon^-(x_0, \lambda) = \frac{1}{\sqrt{1+x_0^2}} - \Phi(x_0, t_-(x_0, \lambda))$$

whose plots are shown in Figures 3.3.1 and 3.3.2 for representative values $\lambda = 0.25$, $\lambda = 0.5$, $\lambda = 0.75$, $\lambda = 1$. The lowest curves correspond to $\lambda = 0.25$ while the highest curves correspond to $\lambda = 1$.

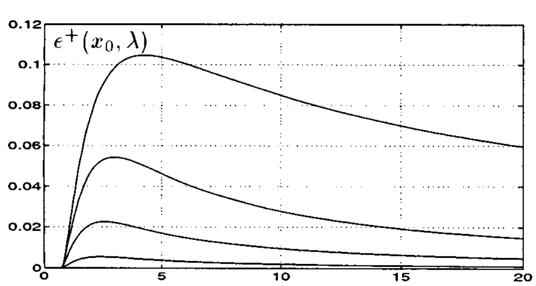

Figure 3.3.1: Plot of Expected Gain Error $\epsilon^+(x_0, \lambda)$

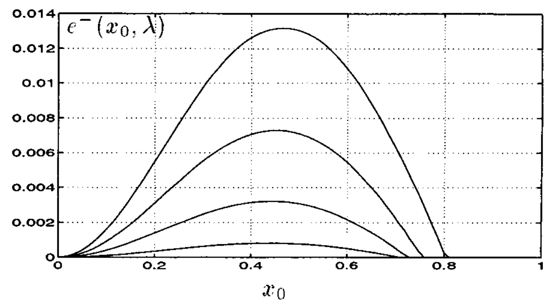

Figure 3.3.2: Plot of Expected Gain Error $\epsilon^-(x_0, \lambda)$

Combining $\epsilon^+(x_0, \lambda)$ and $\epsilon^-(x_0, \lambda)$, the *universal figure of merit* becomes

$$UFOM(\lambda) = \max\left\{ \max_{x_0} \epsilon^+(x_0, \lambda), \max_{x_0} \epsilon^-(x_0, \lambda) \right\}$$

whose plot is shown below.

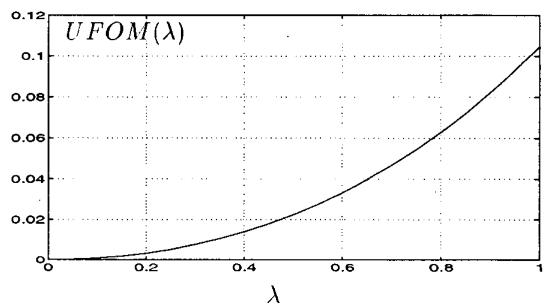

Figure 3.3.3: Plot of Universal Figure of Merit Versus λ

As seen from Figure 3.3.3, for filters of practical interest, the expected filter gain deviates from $g(\omega, \tau_0)$ by small amounts even when the uncertainty scale factor λ is high. For example, with $\lambda = 0.2$ (20% variation in τ), this deviation is seen to be no more than 0.4% of the low frequency gain.

3.4 Remarks: It is also of interest to note that corresponding to τ_0, there is a range of frequency

$$\frac{1}{\sqrt{2}\lambda\tau_0} \leq \omega \leq \frac{0.9811}{\lambda\tau_0}$$

over which the expected gain error satisfies

$$\sup_{f \in \mathcal{F}} |\mathcal{E}g(\omega, \tau^f) - g(\omega, \tau_0)| < 0.014.$$

For the two parameter optimizations involving t and x_0, it is worth noting that there are three distinct regions of x_0 where the errors $\epsilon^-(x_0, \lambda)$ and $\epsilon^+(x_0, \lambda)$ take on different forms. Namely for

$$0 < x_0 \leq \frac{1}{\sqrt{2}}.$$

it can be shown that $\epsilon^+(x_0, \lambda) \equiv 0$ and

$$\epsilon^-(x_0, \lambda) = \frac{1}{\sqrt{1+x_0^2}} - \frac{1}{2\lambda x_0}\{\ln((1+\lambda)x_0 + \sqrt{(1+(1+\lambda)^2 x_0^2)} + \ln((\lambda-1)x_0 + \sqrt{(1+(1-\lambda)^2 x_0^2)}\}.$$

For $\frac{1}{\sqrt{2}} < x_0 < 0.9811$, the previously defined formulae for $\epsilon^-(x_0, \lambda)$ and $\epsilon^+(x_0, \lambda)$ are not obtained analytically. Finally, for $x_0 \geq 0.9811$, it can be shown that $\epsilon^-(x_0, \lambda) \equiv 0$ and

$$\epsilon^+(x_0, \lambda) = -\frac{1}{\sqrt{1+x_0^2}} + \frac{1}{2\lambda x_0}\{\ln((1+\lambda)x_0 + \sqrt{(1+(1+\lambda)^2 x_0^2)} + \ln((\lambda-1)x_0 + \sqrt{(1+(1-\lambda)^2 x_0^2)}\}.$$

These formulae were exploited to facilitate numerical computation associated with Figures 3.3.1, 3.3.2 and 3.3.3.

4. Example: Specific RC Filter

As a specific illustration, we consider a first order filter with nominal $\tau_0 = 0.01$ and parameter tolerance $\lambda = 0.25$. In Figure 4.0.1, we show the resulting maximum expected gain error envelope. This function of frequency is obtained as the maximum of the previously defined lower and upper envelope error measures $\epsilon^-(\omega) \doteq \epsilon^-(\omega\tau_0, \lambda)$ and $\epsilon^+(\omega) \doteq \epsilon^+(\omega\tau_0, \lambda)$.

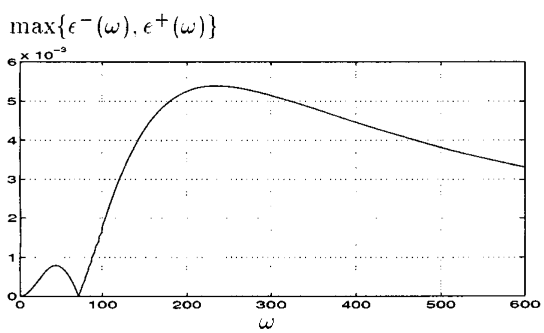

Figure 4.0.1: Example First Order Filter With $\tau_0 = 0.01$

It is noted that with this 25% possible error in τ, the resulting maximum expected gain error is less than 0.6% of the low frequency gain.

5. Conclusions and Possible Generalizations

In this paper, we have calculated various performance measures related to the expected gain of an RC filter. The results were surprising to the extent that the gain errors often turn out to be quite small even when the parameter uncertainty is large. By way of future research, it would be important to consider a more general circuit and study the extent to which similar performance measures can be computed in the presence of multiple uncertainties. To this end, the authors are currently investigating the efficacy of the so-called *spherical uncertainty model* in this context; see [9]. Approaching larger circuits via this approach appears promising because one can avoid the computational difficulties which arise when optimizing multiple truncations associated with the probability distributions for the circuit parameters.

References

[1] Spence, R. and R. S. Soin (1988). "Tolerance Design of Electronic Circuits," Addison-Wesley, New York.

[2] Rubinstein, R. Y. (1981). *Simulation and the Monte Carlo Methods*, Wiley, New York.

[3] Director, S. W. and L. M. Vidigal (1981). "Statistical Circuit Design: A Somewhat Biased Survey," *Proceedings of the European Conference on Circuit Theory Design*, The Hague, The Netherlands, pp. 15-24.

[4] Pinel, J. F. and K. Singhal (1977). "Efficient Monte Carlo Computation of Circuit Yield Using Importance Sampling," *IEEE Proceedings of the International Symposium on Circuits and Systems*," pp. 575-578.

[5] Barmish, B. R. and C. M. Lagoa (1997). "The Uniform Distribution: A Rigorous Justification for its Use in Robustness Analysis," *Mathematics of Control Signals and Systems*, vol. 10, no. 3, pp. 203-222.

[6] Director, S. W. and G. D. Hachtel (1977). "The Simplicial Approximation Approach to Design Centering," *IEEE Transactions on Circuits and Systems*, vol. CAS-24, no. 7, pp. 363-372.

[7] J. Zhang, C. M. Lagoa and B. R. Barmish (1998). "Probabilistic Robustness: An RLC Realization of the Truncation Phenomenon," *Proceeding of the American Control Conference*, Philadelphia.

[8] Lagoa, C. M. (1998). "Contributions to the Theory of Probabilistic Robustness," *Ph.D. Dissertation*, ECE Department, University of Wisconsin-Madison.

[9] B. R. Barmish and C. M. Lagoa and R. Tempo (1997). "Radially Truncated Uniform Distributions for Probabilistic Robustness Analysis," Proceedings of the *American Control Conference*, Albuquerque.