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A Simple Test for Robust Stability of Delay Systems*

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Abstract

Motivated by dynamical system considerations, a number of new results on robust stability of perturbed polynomials have been recently obtained. In the literature under consideration, the perturbations are known only within given bounds and enter affine linearly into the polynomial coefficients. In this paper, this formulation is expanded to include time-delays. Under a properness assumption, a necessary and sufficient condition for robust stability is developed.

1. Introduction and Formulation

A large number of recent papers are motivated by the following scenario: Consider a SISO plant

$$P(s) = \frac{N_P(s)}{D_P(s)} \tag{1}$$

with compensator

$$C(s) = \frac{N_C(s)}{D_C(s)}. \tag{2}$$

Classical stability considerations for this standard closed loop configuration leads to examination of the closed loop polynomial $N_P(s)N_C(s) + D_P(s)D_C(s)$.

In practice, however, the physical parameters entering into the model of a control system may not be known exactly. Typically, these physical parameters are known to vary in prescribed intervals; this leads to a resulting closed loop polynomial which includes perturbations associated with these parameters. We denote these *perturbations* by a vector $q = [\ q_1\, q_2 \ldots q_m\ ]^T$ which is restricted to a prescribed *bounding rectangle* Q in $\mathbf{R}^m$; i.e.,

$$Q = \{q \in \mathbf{R}^m : q_i^- \le q_i \le q_i^+ \quad for\ i = 1, 2, \ldots, m\} \tag{3}$$

where the q_i^- and q_i^+ are specified lower bounds and upper bounds of the i-th perturbation q_i, respectively. In the sequel, the plant numerator and denominator polynomials will be denoted by $N_P(s,q)$ and $D_P(s,q)$ respectively. This leads to a *closed loop polynomial*

$$\delta(s,q) = N_P(s,q)N_C(s) + D_P(s,q)D_C(s). \tag{4}$$

A considerable body of recent research has been aimed at one special case of the formulation above. Namely, it is assumed that the perturbation vector q enters affine linearly into the coefficients of $\delta(s,q)$. In this case, the collection of $\delta(s,q)$ obtained by varying q over Q is a so-called *polytope of polynomials*. Given a desired region $\mathcal{D}$ for the closed loop poles, a fundamental problem is to ascertain whether or not all zeros of $\delta(s,q)$ lie in $\mathcal{D}$ for all $q \in Q$. This being the case, the polytope of polynomials is said to be *$\mathcal{D}$-stable.*

Various special cases of the $\mathcal{D}$-stability problem have been studied in the literature. In the work of Kharitonov (1978), $\delta(s,q)$ corresponds to an interval polynomial and $\mathcal{D}$ is the strict left half plane. Some simplications of Kharitonov's Theorem are provided by Kraus, Anderson, Jury and Mansour (1988) for the case of low order polynomials. In the paper by Bose, Jury and Zeheb (1986), $\delta(s,q)$ is again an interval polynomial and $\mathcal{D}$ is the unit disk. The discrete-time case is also considered for general polytopes in Ackermann and Barmish (1988) and Bartlett and Hollot (1988).

There are also a number of papers providing necessary and sufficient conditions for $\mathcal{D}$-stability with less restrictive assumptions on $\mathcal{D}$ and $\delta(s,q)$. These papers consider a polytope of polynomials with various connectivity assumptions made on $\mathcal{D}$. The Edge Theorem of Bartlett, Hollot and Lin (1988) indicates that a polytope of n-th order polynomials is $\mathcal{D}$-stable if and only if all its exposed edges are $\mathcal{D}$-stable. This result makes it possible to check for $\mathcal{D}$-stability in terms of one parameter; e.g., checking the $\mathcal{D}$-stability of an edge can be accomplished using a root locus plot and in the special case when $\mathcal{D}$ is the strict left half plane, $\mathcal{D}$-stability of an edge can be checked using the eigenvalue test given in Bialas (1985); see also Fu and Barmish (1988) for an extension of the Edge Theorem to handle disconnected $\mathcal{D}$ regions. Another method for checking $\mathcal{D}$-stability of an edge is given by Zeheb (1988). His method involves checking the sign of specially constructed polynomials as a frequency-like variable is swept over some range.

It is pointed out in Barmish (1988), however, that the use of the Edge Theorem often leads to a "combinatoric explosion" in numerical computation. That is, the number of pairwise combinations of extreme polynomials associated with edges "blows up" rapidly as a function of the number of perturbation parameters. For example, there are 2,096,128 pairwise combinations associated with 11 extreme polynomials.

We now mention two approaches in the literature which avoid the combinatonic explosion associated with edges. In the work of Saridereli and Kern (1987), it is seen that the $\mathcal{D}$-stability problem for polytopes of polynomials above can be solved as follows: For each point in the boundary of $\mathcal{D}$, one computes the solution of an appropriately constructed linear program. Subsequently, the infinite set of linear programming solutions are used to ascertain whether or not $\mathcal{D}$-stability is guaranteed; see also Vicino (1988). In Barmish (1988), a boundary sweep is also used but the use of linear programming is totally avoided. Instead, a special *$\mathcal{D}$-stability testing function* $H(\delta)$ is constructed and it is shown that $\mathcal{D}$-stability is guaranteed if and only if $H(\delta) > 0$ for all δ in some bounded precomputable interval.

In view of the existing literature, the main objective of this paper is to incorporate time-delays into the polytope stability framework. Instead of the plant as described above, a special case of the theory in this paper makes it possible handle a plant of the form

$$P_\tau(s,q) = \frac{N_P(s,q)}{D_P(s,q)} e^{-\tau s} \tag{5}$$

where τ is a fixed non-negative delay constant. $\mathcal{D}$-stability considerations lead to examination of the *closed loop uncertain polynomial with time-delay*

$$\delta_\tau(s,q) = N_P(s,q)N_C(s)e^{-\tau s} + D_P(s,q)D_C(s). \tag{6}$$

Notice that when $\tau \to 0$, this formulation degenerates into the one considered in the literature cited above on polytopes of polynomials; i.e., for $\tau = 0$,

$$\delta_\tau(s,q) = \delta(s,q). \tag{7}$$

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In the first part of this paper, we consider only a single *fixed* non-negative delay τ entering into the plant. In Section 4 dealing with extensions, we show how the theory can easily be modified to study $\mathcal{D}$-stability with respect to variations in τ and multiple delays. Within this framework, we generalize the result in Barmish (1988) to the case of delay polynomials; i.e., under appropriate assumptions, we generate a scalar testing function $H_\tau(\delta)$ and show that positivity of $H_\tau(\delta)$ is necessary and sufficient condition for $\mathcal{D}$-stability. It is also important to note that the $\mathcal{D}$-stability theory which we develop in this paper totally avoids the combinatorics associated with edge analysis.

Finally, it should be noted that we are aware of one paper in the literature dealing with a special case of the delay problem above. Namely, in the formulation of Mori and Kokame (1987), $\mathcal{D}$ is taken to be the left half plane and $\delta_\tau(s,q)$ is a polynomial of the form $f(s,e^{-\tau s})+g(s,q)$ where $f(s,e^{-\tau s})$ is a fixed delay polynomial and $g(s,q)$ is an interval polynomial. In Section 4 of this paper, we specialize our main result to obtain the H_τ function for this particular case.[1] It turns out that this strengthening of hypotheses leads to a closed form for H_τ. Moreover, the main "ingredients" going into H_τ are four Kharitonov delay polynomials associated with $f(s,e^{-\tau s})+g(s,q)$. We also indicate how the analysis for this special case is extended to accommodate perturbations in τ.

The paper is organized as follows: In Section 2, we provide the formal notation, definitions and assumptions. In Section 3, we give the main result — a necessary and sufficient condition for the robust stability with delay and a recipe for construction of the testing function H_τ. Section 4 includes a number of extensions and refinements of the main result. Namely, we examine the extent to which our properness assumption can be weakened, generalize the $\mathcal{D}$-stability results to handle perturbed delays and multiple delays, consider a larger class of $\mathcal{D}$ regions and specialize our main result to study the formulation of Mori and Kokame (1987). Section 5 is devoted to the proof of the main result and can be skipped by those readers interested solely in the application. Finally, conclusions are drawn in Section 6.

1.1 A Motivating Technical Remark and Counterexample: From a technical point of view, one fundamental difference between the $\mathcal{D}$-stability problem for ordinary polynomials and the $\mathcal{D}$-stability problem for delay polynomials should be noted: In the literature on ordinary polynomials, it is typically assumed that $\mathcal{D}$ is simply connected, $\delta(s,q)$ has fixed degree and coefficients which depend continuously on q. Subsequently, it is shown that if $\delta(s,q^a)$ is $\mathcal{D}$-stable for some fixed $q^a\in Q$, then $\mathcal{D}$-stability is guaranteed if and only if

$$\delta(s,q)\neq 0 \tag{8}$$

for all s in the boundary of $\mathcal{D}$ and all $q\in Q$.

An important point to note is that this "boundary projection property" does not carry over to delay systems. This point is illustrated by considering the counterexample

$$\delta_\tau(s,q)=(1+q)+(s+1)e^{-s}+(1+q)e^{-2s} \tag{9}$$

where $|q|\leq 1$ and $\mathcal{D}$ is strict left half complex plane. First notice that when $q^a=-1$,

$$\delta_\tau(s,q^a)=(s+1)e^{-s} \tag{10}$$

is strictly stable. Furthermore,

$$\begin{aligned}\delta_\tau(jw,q) &= e^{-jw}[(jw+1)+(q+1)(e^{jw}+e^{-jw})]\\ &= e^{-jw}[(1+2(q+1)\cos w)+jw].\end{aligned}$$

Therefore, it is clear that

$$\delta_\tau(jw,q)\neq 0$$

for all $w\in\mathbf{R}$ and all q with $|q|\leq 1$. On the other hand, when $q^b=1$,

$$\delta_\tau(s,q^b)=2+(s+1)e^{-s}+2e^{-2s} \tag{11}$$

and it is clear that the Nyquist plot of $\delta_\tau(s,q^b)$ encircles the point $-1+j0$ an infinite number of times in the clockwise direction. This implies that $\delta_\tau(s,q^b)$ is unstable.

In order to overcome the type of pathology above, we impose a properness assumption on $\delta_\tau(s,q)$; see Sections 2 and 4.

2. Notation, Definitions and Assumptions

2.1 Notation: In the remainder of this paper, we consider a family of delay polynomials of the form

$$\delta_\tau(s,q)=\delta_0(s,q)+\delta_1(s,q)e^{-\tau_1 s}+\cdots+\delta_N(s,q)e^{-\tau_N s} \tag{12}$$

where $q=[q_1\ q_2\ \cdots\ q_m]^T\in Q$ is the *vector of perturbations*, Q is a given *bounding rectangle* as in (3), $\tau=[\tau_1\ \tau_2\ \cdots\ \tau_N]^T$ is a *vector of delays*, and $\delta_0(s,q),\delta_1(s,q),\ldots,\delta_N(s,q)$ are polynomials whose coefficients are functions of q.

In addition, a desired *zero location region* $\mathcal{D}$ in the complex plane is specified and we let $q^1,q^2,\ldots,q^\ell$ denote extreme points of Q. Using the q^i, we define *the polytope of delay polynomials* Δ_τ consisting of all convex combinations of the *extreme delay polynomials* $\delta_\tau(s,q^1),\delta_\tau(s,q^2),\ldots,\delta_\tau(s,q^\ell)$; i.e.,

$$\Delta_\tau = conv\{\ \delta_\tau(s,q^1),\delta_\tau(s,q^2),\ldots,\delta_\tau(s,q^\ell)\}.$$

Finally, given two complex numbers x and y, we use the inner product notation

$$\langle x,y\rangle \doteq Re\ x\cdot Re\ y+Im\ x\cdot Im\ y.$$

2.2 Definitions: Given any region $\mathcal{D}$ in the complex plane, the polytope of delay polynomials Δ_τ is said to be $\mathcal{D}$*-stable* if for each $q\in Q$, all the zeros of $\delta_\tau(s,q)$ lie in $\mathcal{D}$. For the special case when the region $\mathcal{D}$ is the strict left half plane, for emphasis, we say that Δ_τ is *robustly stable* rather than $\mathcal{D}$-stable.

2.3 Assumptions: The main results in this paper are developed under eight assumptions. The first four assumptions are temporary and will be removed or weakened in Section 4.

Temporary Assumptions:

Assumption 1 (fixed delay): It is assumed that the delay constant τ is fixed.

Assumption 2 (single delay): It is assumed that $N=1$; we write $\tau_1=\tau$.

Assumption 3 (strict properness): It is assumed that

$$deg\ \delta_1(s,q)\ <\ deg\ \delta_0(s,q)$$

for all $q\in Q$.

Assumption 4 (the region $\mathcal{D}$): It is assumed that $\mathcal{D}$ is the strict left half plane.

[1] One minor difference between our specialization and Mori and Kokame's result should be noted: Our formulation includes a properness assumption.

Permanent Assumptions:

Assumption 5 (polytopic structure): It is assumed that the coefficients of $\delta_0(s,q)$ and $\delta_1(s,q)$ are affine linear functions of q.

Assumption 6 (non-triviality): It is assumed that $\delta_\tau(s,q)$ is strictly stable for at least one $q \in Q$. **Assumption 7** (invariant degree of $\delta_0(s,q)$): It is assumed that for some fixed $n \geq 1$,

$$deg\ \delta_0(s,q) = n$$

for all $q \in Q$. For example, this assumption is always satisfied when $\delta_0(s,q)$ is monic.

Assumption 8 (non-negativity of delay): It is assumed that $\tau \geq 0$.

3. Construction of the Robust Stability Testing Function and the Main Result

In this section, there are two main objectives: First, we describe the construction of the *robust stability testing function* H_τ. For the special case when $\tau \to 0$, this function reduces to the one is given in Barmish (1988) for polynomials without time-delays. Having the robust stability testing function in hand, the second main objective is to state the main result — a necessary and sufficient condition for robust stability.

3.1 Construction of the Robust Stability Testing Function $H_\tau(w)$: The starting point for this construction is a region defined as in Barmish (1988). Indeed, let Γ be any region (chosen by the user) in the complex plane $\mathcal{C}$ which has a continuous boundary $\partial\Gamma$ and contains the origin $s = 0$ in its interior. We adopt the notation $\Phi_\Gamma : [0,1] \to \partial\Gamma$ to denote a continuous mapping of a scalar variable ρ onto the boundary of Γ. For example, one simple choice would be to take Γ to be the unit disk and

$$\Phi_\Gamma(\rho) = \cos 2\pi\rho + j \sin 2\pi\rho$$

for $\rho \in [0,1]$. Now, for each fixed pair $(\rho, w) \in [0,1] \times \mathbf{R}$, let

$$h_\tau(\rho, w) \doteq \min_{i \leq \ell} \langle \Phi_\Gamma(\rho), \delta_0(jw, q^i) + \delta_1(jw, q^i) e^{-jw\tau} \rangle. \tag{13}$$

Finally, for each fixed $w \in \mathbf{R}$, define

$$H_\tau(w) \doteq \max_{\rho \in [0,1]} h_\tau(\rho, w). \tag{14}$$

It is important to note that the scalar problem in ρ above only involves a simple computation (for example, a line search over [0,1]). Recalling, however, that the user is free to choose Γ, this calculation can be further trivialized. This is accomplished by taking Γ to be polyhedral; e.g., if

$$\Gamma = \{x + jy \in \mathcal{C} : |x| + |y| \leq 1\},$$

then for fixed w, $h_\tau(\rho, w)$ is a piecewise linear concave function of ρ and the maximum in (14) is immediately obtained. For the special case when $\tau = 0$, numerical computation is discussed in more detail in Barmish (1988).

3.2 The Main Theorem (see Section 5 for proof): *Consider the polytope of delay polynomials Δ_τ satisfying Assumption 1 - 8. Then Δ_τ is robustly stable if and only if*

$$H_\tau(w) > 0 \tag{15}$$

for all $w \in \mathbf{R}$.

3.3 Restriction of the w Sweep to a Bounded Interval: Although the theorem above demands checking positivity condition (15) for all $w \in \mathbf{R}$, the sweep of w can be reduced to a bounded interval. That is, in view of strict properness Assumption 3, there exists a positive frequency w_0 such that for each $q \in Q$ and each $w \in \mathbf{R}$ with $|w| > w_0$,

$$\left|\frac{\delta_1(jw,q)}{\delta_0(jw,q)}\right| < 1. \tag{16}$$

This inequality guarantees that for each $q \in Q$ and each $w \in \mathbf{R}$ with $|w| > w_0$,

$$\delta_\tau(jw,q) \neq 0 \tag{17}$$

which implies that the w-sweep can be restricted to the bounded interval $[-w_0, +w_0]$. An estimate for w_0 can readily be generated using condition (16). To this end, the following steps are used:

Step 1: Generate the upper bound function

$$\delta_1^*(w) \doteq \max_{i \leq \ell} |\delta_1(jw, q^i)| \tag{18}$$

for $|\delta_1(jw,q)|$.

Step 2: Generate the following lower bound for $a_n(q)$, the coefficient of s^n in $\delta_0(s,q)$:

$$a_n^- \doteq \min_{i \leq \ell} |a_n(q^i)|. \tag{19}$$

Note that Assumption 7 guarantees that a_n^- is a positive number.

Step 3: Form the lower bound function

$$\delta_0^*(w) \doteq a_n^- |w^n| - \min_{i \leq \ell} |\delta_0(jw, q^i) - a_n(q^i)(jw)^n| \tag{20}$$

for $|\delta_0(jw,q)|$.

Step 4: Pick w_0 suitably large so that

$$\delta_1^*(w) < \delta_0^*(w) \tag{21}$$

for all w with $|w| > w_0$. Note that Assumption 3 guarantees that w_0 exists.

Steps 1 - 4 above are included for the sake of "completeness of the theory". In practice, however, there is an even simpler way of restricting the w-sweep rather than this estimating w_0. Namely, noting that $H_\tau(w) \to +\infty$ as $w \to \infty$, one can "reasonably" terminate computations when $H_\tau(w)$ is suitably large.

4. Extensions and Refinements

Recall that Assumptions 1 - 4 were designated as "temporary" and were only imposed to simply the exposition. In this section, the objective is to provide extensions and refinements of the main theorem. Subsections 4.1 - 4.4 will consider a sequence of problem formulations, each of which is a generalization of its predecessor. Section 4.5 is devoted to a specialization of our framework to that of Mori and Kokame (1987).

4.1 Weakening of Strict Properness Requirement: For this first level of generalization, we weaken the strict properness requirement. To this end, we replace Assumption 3; i.e., we consider

Assumption 3′ (weak properness): It is assumed that $\delta_\tau(s,q)$ is *weakly proper* in the following sense: For each $q \in Q$,

$$\lim_{w \to \infty} \left| \frac{\delta_1(jw,q)}{\delta_0(jw,q)} \right| < 1. \tag{22}$$

It is clear that strict properness of $\delta_\tau(s,q)$ implies weak properness but not conversely. Now under Assumptions 1, 2, 3′, 4, 5, 6, 7 and 8, it is easily shown that the main theorem still holds. In addition, it is important to note that weak properness is *almost necessary* for robust stability. To see this, observe that the limit of $\frac{\delta_1(jw,q)}{\delta_0(jw,q)}$ always exists as $w \to \infty$ and Δ_τ is not robustly stable if there exists some $q \in Q$ such that

$$\lim_{w\to\infty} \left| \frac{\delta_1(jw,q)}{\delta_0(jw,q)} \right| > 1. \tag{23}$$

This instability is a consequence of the fact that inequality (23) implies that the Nyquist plot for $\frac{\delta_1(s,q)}{\delta_0(s,q)}e^{-\tau s}$ encircles the point $-1+j0$ an infinite number of times in the clockwise direction.

4.2 Other Types of $\mathcal{D}$-Regions: For the next level of generalization, instead of the region $\mathcal{D}$ being the strict left half plane, we allow for other possibilities. One such example is given below.

Assumption 4′ (less restrictive $\mathcal{D}$ regions): The region $\mathcal{D}$ is either the strict left half plane or a finite union of bounded pathwise connected sets.

Now, $\mathcal{D}$-stability analysis can be performed for each pathwise connected component of $\mathcal{D}$. For the k-th such component $\mathcal{D}_k$, we exploit a continuous boundary parameterization $\Phi_{\mathcal{D}_k}(\delta)$ of $\mathcal{D}_k$; i.e., $\Phi_{\mathcal{D}_k}(\delta)$ continuously maps the scalar δ onto the boundary of $\mathcal{D}_k$. Subsequently, the main theorem is modified as follows: For this k-th component, we first define the two variable function

$$h_\tau^k(\rho,\delta) \doteq \min_{i\le\ell}\langle\Phi_\Gamma(\rho), \delta_0(\Phi_{\mathcal{D}_k}(\delta),q^i) + \delta_1(\Phi_{\mathcal{D}_k}(\delta),q^i)e^{-jw\tau}\rangle, \tag{24}$$

and then, for each $\delta\in\mathbf{R}$, we let

$$H_\tau^k(\delta) \doteq \max_{\rho\in[0,1]} h_\tau^k(\rho,\delta). \tag{25}$$

Then, simple modifications of the proof of the main result lead to the following $\mathcal{D}$-stability criterion: Under Assumptions 1, 2, 3′, 4′, 5, 6, 7 and 8, $\mathcal{D}$-stability is guaranteed if and only if

$$H_\tau^k(\delta) > 0 \tag{26}$$

for all k and all $\delta\in\mathbf{R}$.

4.3 Robustness with Respect to a Perturbed Delay: In many applications, the delay constant τ may also undergo perturbation. To obtain the next level of generalization to handle this case, we weaken Assumption 1. Note also that Assumption 6 must also be reinterpreted in this case.

Assumption 1′ (perturbed delay): It is assumed that the delay τ lies within a given interval $\mathcal{T} \doteq [\tau^-,\tau^+]$.

Assumption 6′ (non-triviality): It is assumed that $\delta_\tau(s,q)$ is strictly stable for at least one $q \in Q$ and one $\tau \in \mathcal{T}$.

Now, by modifying the proof in Section 5, under Assumptions 1′, 2, 3′, 4′, 5, 6′, 7 and 8, we arrive at the following conclusion: For fixed $q \in Q$ and perturbed delay τ, $\mathcal{D}$-stability is guaranteed if and only if

$$H_\tau^k(\delta) > 0.$$

for all k, all $\tau \in [\tau^-,\tau^+]$ and all $\delta \in \mathbf{R}$.

Then, in order to include this delay perturbation in the $\mathcal{D}$-stability analysis, one need only perform an additional sweep with respect to the scalar τ; i.e., to test whether $\mathcal{D}$-stability is guaranteed for all $\tau \in [\tau^-,\tau^+]$, we take $H_\tau^k(\delta)$ defined as in (25) and generate

$$H^k(\delta) \doteq \min_{\tau\in[\tau^-,\tau^+]} H_\tau^k(\delta). \tag{27}$$

Subsequently, under Assumptions 1′, 2, 3′, 4′, 5, 6′, 7 and 8, we can easily modify the proof of the main result and conclude that $\mathcal{D}$-stability is guaranteed if and only if

$$H^k(\delta) > 0 \tag{28}$$

for all k and all $\delta \in \mathbf{R}$.

4.4 Extension for Multiple Delays: In this case, we first note that the definition of weak properness needs to be generalized from a single delay to multiple delays; i.e., for $N > 1$, $\delta_\tau(s,q)$ is said to be *weakly proper* if for each $q \in Q$,

$$\lim_{w\to\infty} \sum_{i=1}^{N} \left| \frac{\delta_i(jw,q)}{\delta_0(jw,q)} \right| < 1. \tag{29}$$

With this understanding of weak properness, we retain Assumption 3′ and in addition, Assumption 1′ needs reinterpretation for multiple delays. In this case, we have an interval $[\tau_i^-,\tau_i^+]$ for the i-th delay and we define the *box of delays*

$$\mathcal{T} \doteq \{\ \tau = [\tau_1\ \tau_2\ \cdots\ \tau_N]^T : \tau_i \in [\tau_i^-,\tau_i^+]\ for\ i = 1,2,\ldots,N\ \}. \tag{30}$$

In view of these reinterpretations, Assumption 2 is now eliminated and to generalize the definitions in Subsection 4.3, for the k-th pathwise connected component $\mathcal{D}_k$ of $\mathcal{D}$, we define the two variable function

$$h_\tau^k(\rho,\delta) \doteq \min_{i\le\ell}\langle\Phi_\Gamma(\rho), \delta_0(\Phi_{\mathcal{D}_k}(\delta),q^i) + \sum_{r=1}^{N}\delta_r(\Phi_{\mathcal{D}_k}(\delta),q^i)e^{-jw\tau_r}\rangle. \tag{31}$$

Finally, for each $\delta\in\mathbf{R}$, we take

$$H_\tau^k(\delta) \doteq \max_{\rho\in[0,1]} h_\tau^k(\rho,\delta) \tag{32}$$

and

$$H^k(\delta) \doteq \min_{\tau\in\mathcal{T}} H_\tau^k(\delta). \tag{33}$$

In conclusion, under Assumptions 1′, 3′, 4′, 5, 6′, 7 and 8, $\mathcal{D}$-stability is guaranteed if and only if

$$H^k(\delta) > 0 \tag{34}$$

for all k and all $\delta\in\mathbf{R}$.

It is important to note that from an applications point of view, this refinement is of limited use when the number of perturbed delays is more than 2 or 3. This stems from the fact that the minimization over τ in (33) above is a "vector minimization" rather than a simple scalar minimization as in Subsection 4.3. It is also worth noting that for the important special case when the vector τ is fixed, the extension above can be further simplified; i.e., when $\tau_i^- = \tau_i^+ = \tau_i$ for $i = 1,2,\ldots,N$, instead of (33), one has

$$H^k(\delta) = H_\tau^k(\delta) \tag{35}$$

and computations can proceed quite easily.

Another important special case is obtained when $\tau_i = i\tau$ for $i = 1,2,\ldots,N$. That is, each delay τ_i is an integer multiple of some perturbed delay $\tau \in [\tau^-,\tau^+]$. Hence, we have

$$\delta_\tau(s,q) = \delta_0(s,q) + \sum_{i=1}^{N}\delta_i(s,q)e^{-i\tau s}. \tag{36}$$

and for such a case, the extension of the main result above applies.

4.5 The Interval Delay Polynomial Problem of Mori and Kokame: We now consider the interval delay polynomials studied by Mori and Kokame (1987) as a special case of the framework in this paper. Recalling the discussion in the Introduction, we examine polynomials of the form

$$\delta_\tau(s,q) = f(s,e^{-\tau s}) + g(s,q) \tag{37}$$

where $f(s,e^{-\tau s})$ is a fixed delay polynomial in s and $e^{-\tau s}$ and $g(s,q)$ is an interval polynomial; that is,

$$g(s,q) = \sum_{i=0}^{n}(a_i+q_i)s^i \tag{38}$$

where a_0, a_1, ..., a_n are given, $q \doteq [\ q_0\ q_1\ \cdots\ q_n\ ]^T \in \mathbf{R}^{n+1}$ and $Q = \{q \in \mathbf{R}^{n+1} : q_i^- \leq q_i \leq q_i^+;\ i = 0,1,\ldots,n\}$. Now, we construct the H_τ function. Indeed, it is easily verified using (31) and (32) that with $\mathcal{D}$ being the left half plane, $\delta = w$ and $\Phi_{\mathcal{D}}(w) = jw$, $H_\tau(w)$ is obtained by the following two steps.

Step 1 (Four Kharitonov Delay Polynomials): Form the four Kharitonov polynomials $K_1(s)$, $K_2(s)$, $K_3(s)$ and $K_4(s)$ associated with $g(s,q)$; i.e.,

$$\begin{aligned} K_1(s) &= (a_0+q_0^-)+(a_1+q_1^-)s+(a_2+q_2^+)s^2+(a_3+q_3^+)s^3 \\ &\quad +(a_4+q_4^-)s^4+(a_5+q_5^-)s^5+(a_6+q_6^+)s^6+\cdots; \end{aligned}$$

$$\begin{aligned} K_2(s) &= (a_0+q_0^+)+(a_1+q_1^+)s+(a_2+q_2^-)s^2+(a_3+q_3^-)s^3 \\ &\quad +(a_4+q_4^+)s^4+(a_5+q_5^+)s^5+(a_6+q_6^-)s^6+\cdots; \end{aligned}$$

$$\begin{aligned} K_3(s) &= (a_0+q_0^+)+(a_1+q_1^-)s+(a_2+q_2^-)s^2+(a_3+q_3^+)s^3 \\ &\quad +(a_4+q_4^+)s^4+(a_5+q_5^-)s^5+(a_6+q_6^-)s^6+\cdots; \end{aligned}$$

$$\begin{aligned} K_4(s) &= (a_0+q_0^-)+(a_1+q_1^+)s+(a_2+q_2^+)s^2+(a_3+q_3^-)s^3 \\ &\quad +(a_4+q_4^-)s^4+(a_5+q_5^+)s^5+(a_6+q_6^+)s^6+\cdots \end{aligned}$$

and let $K_1^\tau(s)$, $K_2^\tau(s)$, $K_3^\tau(s)$ and $K_4^\tau(s)$ denote the *four Kharitonov delay polynomials* given by

$$K_1^\tau(s) \doteq K_1(s) + f(s,e^{-s\tau});$$
$$K_2^\tau(s) \doteq K_2(s) + f(s,e^{-s\tau});$$
$$K_3^\tau(s) \doteq K_3(s) + f(s,e^{-s\tau});$$
$$K_4^\tau(s) \doteq K_4(s) + f(s,e^{-s\tau}).$$

Step 2 ($H_\tau(w)$ Function): By a lengthy but straightforward computation, it is readily verified that

$$H_\tau(w) = \begin{cases} max\{ & Re\ K_1^\tau(jw), -Re\ K_2^\tau(jw), \\ & Im\ K_3^\tau(jw), -Im\ K_4^\tau(jw)\} \quad \text{if } w \geq 0, \\ max\{ & Re\ K_1^\tau(jw), -Re\ K_2^\tau(jw), \\ & -Im\ K_3^\tau(jw), Im\ K_4^\tau(jw)\} \quad \text{if } w < 0. \end{cases}$$

In view of the main result, it now follows that the interval delay polynomial (37) is robustly stable if and only if

$$H_\tau(w) > 0 \tag{39}$$

for all $w \in \mathbf{R}$. This is the "frequency domain" version of the Mori and Kokame result. Note that it is easy to extend the result to handle perturbations in the delay; i.e., with

$$H(w) \doteq \min_{\tau\in[\tau^-,\tau^+]} H_\tau(w).$$

the interval delay polynomial is robustly stable for all $\tau \in [\tau^-,\tau^+]$ if and only if

$$H(w) > 0 \tag{40}$$

for all $w \in \mathbf{R}$. For this case of perturbed delay, Mori and Kokame's method would require checking the stability of the $K_i^\tau(s)$ for all $\tau \in [\tau^-,\tau^+]$.

5. Proof of Theorem 3.2

In this section, we give the proof of Theorem 3.2. This is accomplished with the aid of a fundamental proposition involving the behavior of $\delta_\tau(s,q)$ along the imaginary axis.[2] The proof of the proposition below is immediate from the following observations concerning the analyticity of $\delta_\tau(s,q)$.

Observation 5.1: *For fixed τ and q, the distinct zeros of $\delta_\tau(s,q)$ are isolated.*

Observation 5.2: *Using Rouché's Theorem (see Marden (1966)), for fixed τ, the zeros of $\delta_\tau(s,q)$ vary continuously with respect to $q \in Q$; see also Manitius and Olbrot (1979).*

Observation 5.3: *The strict properness of $\delta_\tau(s,q)$ implies that for fixed τ, the right half plane zeros of $\delta_\tau(s,q)$ are uniformly bounded with respect to $q \in Q$.*

In view of these observations, consider a continuous path in q-space beginning at some $q^a \in Q$ such that $\delta_\tau(s,q^a)$ is unstable and terminating at some $q^b \in Q$ such that $\delta_\tau(s,q^b)$ is strictly stable. Then by the observations above, it follows that there is a continuous path of zeros of $\delta_\tau(s,q)$ which crosses the imaginary axis. This result is summarized below.

Proposition 5.4: *$\delta_\tau(s,q)$ is robustly stable if and only if*

$$\delta_\tau(jw,q) \neq 0 \tag{41}$$

for all $w \in \mathbf{R}$ and all $q \in Q$.

Now we are in position to give a proof of the main theorem.

Proof of Theorem 3.2: Having established Proposition 5.4, the line of reasoning closely follows that given in Barmish (1988). For each fixed $w \in \mathbf{R}$, we define the *value set* in the complex plane; i.e., let

$$\begin{aligned} \Delta_\tau(w) &\doteq \{\delta_\tau(jw,q) : q \in Q\} \\ &= conv\{\delta_\tau(jw,q^1), \delta_\tau(jw,q^2), \ldots, \delta_\tau(jw,q^\ell)\}. \end{aligned}$$

Observe that $\Delta_\tau(w)$ is the polytope in the complex plane $\mathcal{C}$ and moreover, by Proposition 5.4, Δ_τ is robustly stable if and only if for each $w \in \mathbf{R}$,

$$0 \notin \Delta_\tau(w).$$

That is, Δ_τ is robustly stable if and only if for each $w \in \mathbf{R}$, the origin in the complex plane $\mathcal{C}$ can be separated from the polytope $\Delta_\tau(w)$ by a line. Equivalently, Δ_τ is robustly stable if and only if for each $w \in \mathbf{R}$, there exists some non-zero vector $\eta \doteq [\eta_1\ \eta_2]^T$ such that

$$\eta_1 Re\ \delta_\tau(jw,q) + \eta_2 Im\ \delta_\tau(jw,q) > 0 \tag{42}$$

for all $q \in Q$.

Furthermore, since the region Γ includes zero as an interior point, without loss of generality, the "search" for η above can be restricted to the boundary $\partial\Gamma$. Next, we note that the separation condition above need only be checked at the extreme points of $\Delta_\tau(w)$. Hence, it follows that Δ_τ is robustly stable if and only if for each $w \in \mathbf{R}$, there exists some $\eta \in \partial\Gamma$ such that

[2] the authors express gratitute to Professor A. Olbrot for discussion of this point.

$$\eta_1 Re\ \delta_\tau(jw, q^i) + \eta_2 Im\ \delta_\tau(jw, q^i) > 0 \qquad (43)$$

for $i \in \{1, 2, \ldots, \ell\}$. Equivalently, Δ_τ is robustly stable if and only if for each $w \in \mathbf{R}$, there exists some $\rho \in [0, 1]$ such that

$$Re\ \Phi_\Gamma(\rho) \cdot Re\ \delta_\tau(jw, q^i) + Im\ \Phi_\Gamma(\rho) \cdot Im\ \delta_\tau(jw, q^i) > 0 \qquad (44)$$

for $i \in \{1, 2, \ldots, \ell\}$. Substituting for $h_\tau(\rho, w)$ above, we arrive at the following point: Δ_τ is robustly stable if and only if for each $w \in \mathbf{R}$, there exists some $\rho \in [0, 1]$ such that

$$h_\tau(\rho, w) > 0. \qquad (45)$$

The proof is now completed by re-expressing this condition in terms of the maximum of $h_\tau(\rho, w)$ with respect to $\rho \in [0, 1]$. That is, Δ_τ is robustly stable if and only if for each $w \in R$,

$$\begin{aligned} H_\tau(w) &= \max_{\rho \in [0,1]} h(\rho, w) \\ &> 0. \end{aligned}$$

6. Conclusion

In this paper, a necessary and sufficient condition was given for $\mathcal{D}$-stability of a polytope of delay polynomials. Perhaps the most restrictive assumption in this work and the cited literature is the affine linearity requirement on the coefficient perturbations. Although this is a step up from the independent coefficient assumption of Kharitonov (1978), it still does not capture all systems of interest. More generally, one wishes to handle systems whose coefficients are polynomic functions of the perturbations. In order to apply the results of this paper to those problems, one must resort to overbounding the coefficients by affine linear functions. Although this can always be done, such a procedure introduces conservatism into the analysis.

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