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The Robust Root Locus

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Abstract

In this paper, a simple technique is described for generating the root loci of a feedback system which includes perturbations $q \in \mathbf{R}^l$ entering affine linearly into the coefficients of the plant. Denoting the perturbed plant by $\mathcal{P}(s,q)$ and the compensator by $C(s)$, we address the following problem: Given a bounding set $Q \subset \mathbf{R}^l$ for q, find the locus of points z in the complex plane such that $1 + k\mathcal{P}(z,q)C(z) = 0$ for some $k \geq 0$ and some $q \in Q$. Such points z are said to lie on the *Robust Root Locus*. One of the strengths of the technique presented in this paper is that it avoids the "combinatoric explosion" synonomous with gridding the l-dimensional set Q and plotting a large number of ordinary root loci associated with the grid the points. Instead, the technique given here exploits only a 2-dimensional gridding of a bounded subset of the complex plane.

1. Introduction and Formulation

To motivate this paper, recall the classical Root Locus Problem: Given a nominal plant transfer function $P(s)$ and a fixed compensator $C(s)$, find the locus of points $s \in \mathbf{C}$ such that

$$1 + kP(s)C(s) = 0 \qquad (1)$$

for some $k \geq 0$. That is, find the locus of the closed loop pole locations as a function of the gain k; e.g., see Krall (1970) for a survey of classical root locus techniques.

The takeoff point for this paper is the following fact: When performing robustness analysis and design, various physical parameters of the plant may be perturbed. Hence, the "true" root locus may differ from the root locus generated using the so-called nominal system. In this regard, the main focal point of this paper is the generation of root loci with respect to variations in more than one parameter. This work can be viewed as a systematic approach to robustness problems which were alluded to in the fifties and sixties; e.g., see Truxal (1955). In the later work by Zeheb and Walach (1977), a two-parameter root locus problem is considered and rather specific assumptions (motivated by circuit theory) are made about the class of perturbations.

A number of papers following Zeheb and Walach's work deal with the so-called *zero set concept*; e.g., see Zeheb and Walach (1981) and Fruchter, Srebro and Zeheb (1987). It is seen that the zero set provides a rather general framework for dealing with multi-parameter root-loci. That is, for each value of the gain k, a function can be given whose zero set corresponds with the cross section of the root locus at k. Hence, in principle, a multi-parameter root locus can be generated by computing these cross sections as k varies. In practice, however, there are difficulties associated with numerical computation of the zero set.

By strengthening the hypotheses on the class of allowable perturbations, we obtain an enormous simplification in the characterization of multi-parameter root loci. To this end, attention in this paper is restricted to the case when the perturbations enter the plant coefficients either linearly or multilinearly. This leads to the following type of result: A point $z \in \mathbf{C}$ can be "tested" for membership in the root locus by simply checking whether or not the point $s = 0$ lies inside or outside a specially constructed 2-dimensional convex polygon in the complex plane. Hence, one avoids the "combinatoric explosion" in computation associated with an l-dimensional gridding of the rectangle Q.

This type of "zero inclusion" condition goes back a long way in the robust stability literature (see, for example, Zadeh and Desoer (1963)) and has been revived in the last few years; e.g., see Saeki (1986), de Gaston and Safonov (1988), Dasgupta (1988), Minnichelli, Anagnost and Desoer (1988). Furthermore, this condition is easy to check because only two dimensions are involved and the set to be checked is a convex polygon. The first distinguishing feature of this paper is the use of the zero inclusion condition within the context of root locus rather than robust stability as in previous work. The second distinguishing feature is that it readily lends itself to practical application and computed-aided analysis and design.

To study variations in the root loci under multiple parameter variations, we let $q \in \mathbf{R}^l$ denote a *vector of perturbations* with i-th component q_i satisfying

$$q_i^- \leq q_i \leq q_i^+ \qquad (2)$$

where the q_i^+ and q_i^- are prescribed bounds. Hence, the vector of perturbations q is confined to the l-dimensional rectangle

$$Q = \left\{ q : q_i^- \leq q_i \leq q_i^+ ;\ i = 1, \ldots, l \right\}. \qquad (3)$$

To denote the dependence of the plant transfer function on q, we write $\mathcal{P}(s,q)$ and when $q = 0$, we obtain the so-called *nominal plant*; i.e.,

$$P(s) = \mathcal{P}(s,0). \qquad (4)$$

We now present the formal definition of the Robust Root Locus (RRL): A point $z \in \mathbf{C}$ is said to lie on the *Robust Root Locus* (RRL) if

$$1 + k\mathcal{P}(z,q)C(z) = 0 \qquad (5)$$

for some $k \geq 0$ and some $q \in Q$. It is also of interest to study the distribution of the closed loop poles for a fixed value of the gain k. Hence, for fixed $k = k^*$, a point $z \in \mathbf{C}$ is said to lie on the *cross section of the* RRL *at* k^* if

$$1 + k^*\mathcal{P}(z,q)C(z) = 0 \qquad (6)$$

for some $q \in Q$.

The first main objective of this paper is to provide a computationally tractable method to accurately generate the RRL for the case when the coefficients of the plant are affine linear functions of q. We do not simply overbound the distribution of the poles in the s-plane — we find the distribution precisely. This lack of conservatism may seem somewhat surprising because of the complicated relationship between the zeros of a polynomial and its coefficients; e.g., see Marden (1966).

In a sense, this paper can be viewed as a complement to the growing body of literature on parameter space space methods; see e.g., Ackermann (1980). One reason why parameter space methods are powerful is because it is much easier to study the effect of feedback on the coefficients of the characteristic polynomial than on the closed loop poles. One of the earliest papers to fully recognize and exploit this fact is by Fam and Meditch (1978). This is also recently recognized in Soh, Evans, Petersen and Betz (1987) and Biernacki, Huang and Bhattacharyya (1987).

Noting, however, that the closed loop poles may be highly sensitive to the coefficients of the characteristic polynomial, a design which "looks good" from a parameter space point of view may have poles which are badly "smeared" over a large region in the complex plane. Hence, having the RRL available, one has the opportunity to check whether or not this phenomenon is occuring; i.e., examine the cross section of the RRL at $k = 1$. Subsequently, by tuning the loop gain, there is an opportunity to obtain a more desirable distribution of closed loop poles; i.e., the RRL can be used for robustness enhancement.

The remainder of this paper is organized as follows: In Section 2, we introduce the necessary notation for the parameterization of the closed loop system with respect to $q \in Q$. Section 3 provides a simple technique to decide whether a given point $z \in \mathbf{C}$ lies on the RRL. In Section 4, the RRL technique is reinterpreted in terms of the control system parameters and in Section 5, a crude bound for the cross sections of the RRL is computed. Finally, in Section 6 we present a numerical example and in Section 7, we provide conclusions.

2. Basic Notation

2.1 The Extreme Points of the Rectangle Q: As stated in Section 1, we assume that the vector of perturbations q is confined to the l-dimensional rectangle Q. This rectangle has at most $L \doteq 2^l$ extreme points denoted $q^1, q^2, \ldots, q^L$. The i-th extreme point has j-th component given by

$$q_j^i = q_j^- \text{ or } q_j^i = q_j^+.$$

2.2 The Plant Parameterization: For each fixed $q \in Q$, the plant is a proper single-input single-output system[1] represented by the proper rational transfer function

$$\mathcal{P}(s,q) = \frac{N_P(s,q)}{D_P(s,q)} \tag{7}$$

where $N_P(s,q)$ and $D_P(s,q)$ are polynomials in s of the form

$$\begin{aligned} N_P(s,q) &= a_{n_P}(q)s^{n_P} + a_{n_P-1}(q)s^{n_P-1} + \ldots + a_1(q)s + a_0(q); \\ D_P(s,q) &= s^{d_P} + b_{d_P-1}(q)s^{d_P-1} + \ldots + b_1(q)s + b_0(q). \end{aligned} \tag{8}$$

Furthermore, it is assumed that the coefficients $a_i(q)$ and $b_i(q)$ above are given affine linear functions of the perturbation vector q; i.e.,

[1]The development to follow readily extends to multi-input single-output systems and to single-input multi-output systems.

$$a_i(q) = a_{i0} + \sum_{j=1}^{l} a_{ij} q_j$$

for $i = 0, 1, \ldots, n_P$ and

$$b_i(q) = b_{i0} + \sum_{j=1}^{l} b_{ij} q_j$$

for $i = 0, 1, \ldots, d_P - 1$. From Section 1 recall that the nominal plant (4) is given by $P(s) = \mathcal{P}(s, 0)$.

2.3 The Compensator: As usual, the compensator to be applied to the plant $\mathcal{P}(s,q)$ is a proper rational transfer function

$$C(s) = k\,\frac{N_C(s)}{D_C(s)} \tag{9}$$

where $k \geq 0$ is an *adjustable gain* and

$$\begin{aligned} N_C(s) &= \alpha_{n_C}s^{n_C} + \alpha_{n_C-1}s^{n_C-1} + \ldots + \alpha_1 s + \alpha_0 \\ D_C(s) &= s^{d_C} + \beta_{d_C-1}s^{d_C-1} + \ldots + \beta_1 s + \beta_0 \end{aligned} \tag{10}$$

are prescribed polynomials.

2.4 Vector Representation of The Closed Loop Equation: Using the notation introduced above, the *closed loop equation* is given by

$$\Delta(s,q,k) = kN_P(s,q)N_C(s) + D_P(s,q)D_C(s). \tag{11}$$

We introduce the vector $\delta(q,k) \in \mathbf{R}^{d_C+d_P}$ to represent the coefficients for $\Delta(s,q,k)$; i.e, $\delta(q,k)$ has i-th component $\delta_{i-1}(q,k)$ and hence, we write

$$\Delta(s,q,k) = s^{d_C+d_P} + \sum_{i=0}^{d_P+d_C-1} \delta_i(q,k)s^i. \tag{12}$$

In Section 4, we express the vector $\delta(q,k)$ more directly in terms of the coefficients a_{ij}, b_{ij}, α_i and β_i describing the plant $\mathcal{P}(s,q)$ and the compensator $C(s)$. This type of expression proves useful for numerical computation.

3. Key Ideas Behind the Construction of the RRL

As emphasized in the Introduction, it is typically not feasible to generate an RRL by gridding the l-dimensional set Q. In this section, we see that the computation of the RRL can be achieved using only a 2-dimensional gridding of a bounded subset of the complex plane. This reduction in complexity leads to computational tractability which will be demonstrated using the numerical example presented in Section 7.

3.1 Key Ideas for the Special Case of an Interval Plant with a Static Gain: To motivate the general construction of the RRL, we consider the special case of interval polynomials for $N_P(s,q)$ and $D_P(s,q)$ and a static gain compensator

$$C(s) = k\,. \tag{13}$$

In this case, each plant coefficient depends on *one and only one* perturbation parameter q_i; say

$$a_i(q) = a_{i0} + q_{i+1}$$

for $i = 0, 1, \ldots, n_P$ and

$$b_i(q) = b_{i0} + q_{n_P+i+2}$$

for $i = 0, 1, \ldots, d_{P-1}$. Then, it is easily verified that the closed loop polynomial is given by

$$\Delta(s,q,k) = s^{n_C} + \sum_{i=0}^{n_P} (ka_{i0} + b_{i0} + kq_{i+1} + q_{n_P+i+2})\, s^i + \sum_{i=n_P+1}^{n_C-1} (b_{i0} + q_{n_P+i+2}) \, . \quad (14)$$

Hence, we see that $\Delta(s,q,k)$ is an interval polynomial; i.e., we can write

$$\Delta(s,q,k) = s^{n_C} + \sum_{i=0}^{n_C-1} \delta_i(q,k)s^i \quad (15)$$

where

$$\delta_i(q,k) \in [\delta_i^-(k), \delta_i^+(k)] \quad (16)$$

for $i = 0, 1, \ldots, n_{C-1}$ and

$$\delta_i^-(k) = \begin{cases} ka_{i0} + b_{i0} + kq_{i+1}^- + q_{n_P+i+2}^- & \text{for } i = 0,1,\ldots,n_P; \\ b_{i0} + q_{n_P+i+2}^- & \text{for } i = n_{P+1},\ldots,n_{C-1} \end{cases} \quad (17)$$

and

$$\delta_i^+(k) = \begin{cases} ka_{i0} + b_{i0} + kq_{i+1}^+ + q_{n_P+i+2}^+ & \text{for } i = 0,1,\ldots,n_P; \\ b_{i0} + q_{n_P+i+2}^+ & \text{for } i = n_{P+1},\ldots,n_{C-1} \, . \end{cases} \quad (18)$$

We now motivate the construction of the cross section of the RRL at $k = 1$. In this regard, we use the well-known Kharitonov polynomials to determine which points on the imaginary axis lie on this cross section. The idea used to characterize these axis points is generalized to "non-axis points" and to "non-interval polynomials" in the next subsection.

Indeed, for fixed frequency $\omega^* \in \mathbf{R}$ and fixed gain $k = k^*$, we want to decide if $s = j\omega^*$ lies on the cross section of RRL at k^*. To this end, first note that the work of Kharitonov (1978) leads to the sharpest possible bounds on $\Delta(j\omega^*, q, k^*)$. Namely, we have

$$\begin{aligned} Re\ K_1(j\omega^*,k^*) \le\ & Re\ \ \Delta(j\omega^*,q,k^*) \le Re\ K_2(j\omega^*,k^*); \\ Im\ K_3(j\omega^*,k^*) \le\ & Re\ \ \Delta(j\omega^*,q,k^*) \le Im\ K_4(j\omega^*,k^*) \end{aligned} \quad (19)$$

where

$$\begin{aligned} K_1(s,k) &= \delta_0^-(k) + \delta_1^-(k)s + \delta_2^+(k)s^2 + \delta_3^+(k)s^3 + \cdots; \\ K_2(s,k) &= \delta_0^+(k) + \delta_1^+(k)s + \delta_2^-(k)s^2 + \delta_3^-(k)s^3 + \cdots; \\ K_3(s,k) &= \delta_0^+(k) + \delta_1^-(k)s + \delta_2^-(k)s^2 + \delta_3^+(k)s^3 + \cdots; \\ K_4(s,k) &= \delta_0^-(k) + \delta_1^+(k)s + \delta_2^+(k)s^2 + \delta_3^-(k)s^3 + \cdots \end{aligned} \quad (20)$$

are the four Kharitonov polynomials.

Denoting the rectangle associated with (19) by

$$\Omega(\omega^*,k^*) \doteq \left\{\Delta(j\omega^*,q,k^*) : q_i^- \le q_i \le q_i^+ \ \text{ for } \ i = 1,2,\ldots,l\right\}, \quad (21)$$

observe the sharpness of the bounds in (19) implies that all points in $\Omega(\omega^*,k^*)$ are attainable by choice of q. Hence, it follows that $s = j\omega^*$ belongs to the cross section of the RRL at k^* if and only if

$$0 \in \Omega(\omega^*,k^*) \quad (22)$$

or, equivalently,

$$\begin{aligned} Re\ K_1(j\omega^*,k^*) \le\ \ 0\ \ & \le Re\ K_2(j\omega^*,k^*) \\ Im\ K_3(j\omega^*,k^*) \le\ \ 0\ \ & \le Im\ K_4(j\omega^*,k^*). \end{aligned} \quad (23)$$

Hence, we arrive at the following conclusion: The point $s = j\omega^*$ with $\omega^* \ge 0$ belongs to the cross section of the RRL at k^* if and only if (23) holds. It is easy to verify that for $\omega^* \le 0$, the identical condition holds with the labels reversed for $K_1(j\omega, k)$ and $K_2(j\omega, k)$.

Observation: An equivalent way to write (23) is

$$H(\omega^*,k^*) \le 0 \quad (24)$$

where

$$H(\omega,k) \doteq \max\{K_1(j\omega,k), -K_2(j\omega,k), K_3(j\omega,k), -K_4(j\omega,k)\}. \quad (25)$$

This way of checking the zero inclusion condition motivate a method for performing computations in the more general analysis to follow.

3.2 Key Ideas for the General Case: Now we consider the more general case obtained when the plant coefficients depend affine linearly on q and we want to determine whether a given point $s = z$ belongs to the RRL. Motivated by the interval polynomial analysis of the previous subsection, we fix $k = k^*$ and define the set

$$\Omega(z,k^*) \doteq \{\Delta(z,q,k^*) : q \in Q\}\,. \quad (26)$$

It now follows that z belongs to the cross section of the RRL at k^* if and only if

$$0 \in \Omega(z,k^*)\ . \quad (27)$$

Finally, to test zero inclusion condition above, we need a more concrete description of $\Omega(z,k^*)$. To obtain such a description, we first note that an affine linear transformation T taking the rectangle Q into the complex plane has a convex polygonal image which is the convex hull of T applied to the extreme points q^i; i.e.,

$$T(Q) = \text{conv}\left\{T(q^1), T(q^2), \ldots, T(q^L)\right\}\ . \quad (28)$$

Hence, if we consider T to be an affine linear mapping, taking q into $\delta(q,k^*)$, it follows that $\Omega(z,k^*)$ is the convex polygon descibed by

$$\Omega(z,k^*) = \text{conv}\left\{\Delta(z,q^1,k^*), \Delta(z,q^2,k^*), \ldots, \Delta(z,q^L,k^*)\right\}. \quad (29)$$

Book Keeping Simplification: Note that the zero inclusion condition (27) can be checked in many possible ways because $\Omega(z,k^*)$ is a convex polygon—even a brute force gridding procedure will suffice. Nevertheless, when performing digital computation, it is convenient to have a "compact formula" which can easily be coded. Recalling inequality (24) for interval polynomials with a static gain compensator, we now provide a similar result for this affine linear case; i.e., we generate a function $H(z,k)$ having the following property (Proved in Barmish and Tempo (1988)): A point z belongs to the cross section of the RRL at k^* if and only if

$$H(z,k^*) \leq 0 \ . \tag{30}$$

We now provide a recipe for $H(z,k^*)$.

Step 1: Pick any continuous path Γ in the complex plane which encircles the origin and let $\Phi_\Gamma(\rho)$ denote a parameterization of this path which is obtained by varying ρ between 0 and 1; i.e.,

$$\Gamma \doteq \{\Phi_\Gamma(\rho) : \rho \in [0,1]\} \ . \tag{31}$$

A simple example of an acceptable path is described by the boundary of the unit circle given by

$$\Phi_\Gamma(\rho) = \cos\ 2\pi\rho + j \sin\ 2\pi\rho \ . \tag{32}$$

More generally, for zero inclusion problems, in Barmish (1988) it is argued that computations are facilited when Γ is a polyhedral set. For example, one reasonable choice is the unit "diamond" described by

$$|Re\ s| + |Im\ s| = 1. \tag{33}$$

Step 2: Define the function

$$h(z,k^*,\rho) \doteq \min_{i \leq L} < \Phi_\Gamma(\rho), \Delta(z,q^i,k^*) > \tag{34}$$

where for $z_1,\ z_2 \in \mathbf{C}$, we use the inner product

$$< z_1, z_2 > \doteq Re\ z_1 \cdot Re\ z_2 + Im\ z_1 \cdot Im\ z_2 \ . \tag{35}$$

Step 3: Let

$$H(z,k^*) \doteq \max_{0 \leq \rho \leq 1} h(z,k^*,\rho). \tag{36}$$

Notice that numerical generation of this function involves a sweep of the scalar variable ρ over the interval $[0,1]$.

4. A More Concrete Formula for $\delta(q,k)$

The computations required in Section 3 can be further simplified by taking advantage of the underlying control system structure. In this section, we derive an expression for the closed loop vector $\delta(q,k)$ in terms of the control system parameters. The computation below essentially amounts to a minor modification of the idea used in Biernacki, Huang and Bhattacharyya (1987) to isolate the effect of the gain k on the vector $\delta(q,k)$. Note that this leads to computational savings because various matrices need only be computed once as k varies along the RRL. Indeed, for notational convenience, we define

$$\alpha_i \doteq 0$$

for $i > n_C$. Finally, we define the *compensator parameter matrices* as follows:

$$\Theta_\alpha \doteq \begin{bmatrix} \alpha_0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ \alpha_1 & 0 & \alpha_0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ \alpha_2 & 0 & \alpha_1 & 0 & \cdot & \cdot & 0 & 0 & 0 \\ \alpha_3 & 0 & \alpha_2 & 0 & \cdot & \cdot & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \alpha_{d_C-1} & 0 & \alpha_{d_C-2} & 0 \\ 0 & 0 & 0 & 0 & \cdot & \alpha_{d_C} & 0 & \alpha_{d_C-1} & 0 \end{bmatrix} \in \mathbf{R}^{(d_C+d_P)\times 2(d_P+1)} \tag{37}$$

and

$$\Theta_\beta \doteq \begin{bmatrix} 0 & \beta_0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & \beta_1 & 0 & \beta_0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & \beta_2 & 0 & \beta_1 & \cdot & 0 & \cdot & 0 & 0 \\ 0 & \beta_3 & 0 & \beta_2 & \cdot & 0 & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & 0 & \beta_{d_C-1} & 0 & \beta_{d_C-2} \\ 0 & 0 & 0 & 0 & \cdot & 0 & 1 & 0 & \beta_{d_C-1} \end{bmatrix} \in \mathbf{R}^{(d_C+d_P)\times 2(d_P+1)}. \tag{38}$$

Similarly, to describe the plant perturbations, we let

$$a_{ij} \doteq 0$$

for $i > n_P$ and $j = 0,1,\ldots,l$. Subsequently, we define the *perturbation coefficient matrix* as

$$\Theta_{ab} \doteq \begin{bmatrix} a_{01} & a_{02} & \ldots & a_{0l} \\ b_{01} & b_{02} & \ldots & b_{0l} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d_P1} & a_{d_P2} & \ldots & a_{d_Pl} \\ 0 & 0 & \ldots & 0 \end{bmatrix} \in \mathbf{R}^{2(d_P+1)\times l} \tag{39}$$

and the *nominal plant coefficient vector* as

$$\theta_0 \doteq \begin{bmatrix} a_{00} \\ b_{00} \\ \vdots \\ a_{d_P0} \\ 1 \end{bmatrix} \in \mathbf{R}^{2(d_P+1)}. \tag{40}$$

The following lemma is obtained by a lengthy but straightforward computation.

4.1 Lemma (See Barmish and Tempo (1988) for proof): *Let Θ_α, Θ_β, Θ_{ab} and θ_0 be defined as in (37), (38), (39) and (40) respectively. Then*

$$\delta(q,k) = (k\Theta_\alpha + \Theta_\beta)(\theta_0 + \Theta_{ab}q)\,. \tag{41}$$

Remark: To exploit the function $H(z,k)$ in checking if a point z belongs to the RRL, one simply substitutes (41), into (34) and (36).

5. Simplification of RRL Computation using a Crude Bound

Note that if $k = k^*$ is fixed, we have a crude bound $\mathcal{D}_{k^*}$ for the cross section of the RRL at k^*. Hence, one can avoid testing zero inclusion condition for an unbounded set of z; i.e., we generate a 2-dimensional grid of $\mathcal{D}_{k^*}$ and check if zero belongs to $\Omega(z,k^*)$ as z ranges over $\mathcal{D}_{k^*}$.

Indeed, for $k \geq 0$ and $s \in \mathbf{C}$ we seek a bound $\mathcal{D}_k$ on the zeroes of $\Delta(s,q,k)$; i.e.,

$$\{s \in \mathbf{C} : \Delta(s,q,k) = 0 \ \text{ for some } \ q \in Q\} \subseteq \mathcal{D}_k. \tag{42}$$

To generate an acceptable bound $\mathcal{D}_k$, we use the well known fact (see Marden (1966)) that for a fixed monic n-th order polynomial $p(s) = s^n + \sum_{i=0}^{n-1} a_i s^i$, all its zeros are interior to the circle of radius

$$R = 1 + \max\{|a_i| : i = 0, 1, \ldots, n-1\}. \quad (43)$$

Applying this result to equation (12), it follows that all zeros of $\Delta(s, q, k)$ lie interior to a circle of radius

$$R_k = 1 + \max_{i,j} |\delta_i(q^j, k)|. \quad (44)$$

To complete the bounding process, we substitute for $\delta_i(q^j, k)$ in terms of the given plant and compensator coefficient functions. Indeed, letting e_i denote a unit vector in the i-th coordinate direction and (41), we obtain

$$R_k = 1 + \max_{i,j} |e_i^T (k\Theta_\alpha + \Theta_\beta) \left(\theta_0 + \Theta_{ab} q^j\right) | \quad (45)$$

and the desired bounding region is

$$\mathcal{D}_k \doteq \left\{ s \in \mathbf{C} : (Re\ s)^2 + (Im\ s)^2 \leq R_k^2 \right\} . \quad (46)$$

Note that even greater computational savings can be achieved by using tighter bounds for the zeros of $\Delta(s, q, k)$; e.g., see Marden (1966).

6. Numerical Example

In this Section, we illustrate the application of the RRL technique using a plant with transfer function

$$\mathcal{P}(s, q) = \frac{1}{s(s^2 + (8 + q_1)s + (20 + q_2))}, \quad (47)$$

perturbation bounds

$$-2 \leq q_1 \leq 2,$$
$$-4 \leq q_2 \leq 4$$

and unity feedback.

Computational Results: The RRL was generated for the range

$$0 \leq k \leq 100. \quad (48)$$

A few observations are in order: For the low gain case ($k < 23$), it is apparent from the cross section of the RRL that one cannot distinguish between the variations of the three closed loop poles. This phenomenon is illustrated in Figure 1 for $k = 12$. On the other hand, for the high gain case ($k \geq 23$), there are three distinct regions–two regions corresponding to a pair of complex conjugate poles and an interval corresponding to a real pole. Typical cross sections of the RRL are given in Figures 2 and 3 for $k = 25$ and $k = 50$ respectively.

7. Conclusion

One of the main technical novelties of this paper is the use of the zero inclusion conclusion within the context of robust root locus. To date, this condition has been used largely in a robust stability context. When the RRL is recast in this framework, a 2-dimensional sweep of a bounded set is required in lieu of the one dimensional sweep used in robust stability theory. As illustrated by the numerical example, the resulting computational technique is easy to use and readily lends itself to useful graphical display.

A second important point to note is that for affine linear uncertainty structures, we obtain the "true" RRL. That is, at each value of the gain k, one obtains (within the limits of numerical roundoff) the exact distribution of the closed loop poles.

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References

Ackermann, J.E. (1980). Parameter Space Design of Robust Control Systems, *IEEE Transactions on Automatic Control*, AC-25, pp. 1058-1072.

Barmish, B.R. (1988). A Generalization of Kharitonov's Four Polynomial Concept for Robust Stability Problems with Linearly Dependent Coefficient Perturbations, *Proceedings of the American Control Conference*, Atlanta.

Barmish, B.R. and R. Tempo (1988). The Robust Root Locus, *Technical Report CENS-CNR, Politecnico di Torino*.

Biernacki, R.M., H. Hwang and S.P. Bhattacharyya (1987). Robust Stability with Structured Real Parameter Perturbations, *IEEE Transactions on Automatic Control*, AC-32, pp. 495-506.

Dasgupta, S. (1988). Kharitonov's Theorem Revisited, to appear in *Systems and Control Letters*.

de Gaston, R.R.E. and M.G. Safonov (1988). Exact Calculation of the Multiloop Stability Margin, *IEEE Transactions on Automatic Control*, AC-33, pp. 156-171.

Fam, A.T. and J.S. Meditch (1978). A Canonical Parameter Space for Linear Systems, *IEEE Transactions on Automatic Control*, AC-23, pp. 454-458.

Fruchter, G., U. Srebro and E. Zeheb (1987). On several Variable Zero Set and Application to MIMO Robust Feedback Stabilization, *IEEE Transactions on Circuits and Systems*, CAS-34, pp. 1210-1220.

Kharitonov, V.L. (1978). Asymptotic Stability of an Equilibrium Position of a Family of Systems of Linear Differential Equations, *Differential'nye Uraveniya*, 14, pp. 1483-1485.

Krall, A.M. (1970). The Root Locus Method: A Survey, *SIAM Review*, 12, pp. 64-72.

Marden, M. (1966). Geometry of Polynomials, *Mathematical Surveys*, no. 3, American Mathematical Society, Providence.

Minnichelli, R.J., J.J. Anagnost and C.A. Desoer (1988). An Elementary Proof of Kharitonov's Stability Theorem with Extensions, to appear.

Soh, Y.C., R.J. Evans, I.R. Petersen, and R.E. Betz (1987). Robust Pole Assignment, *Automatica*, 23, pp. 601-610.

Truxal, J.G. (1955). *Automatic Feedback Control System Synthesis*, McGraw-Hill, New York.

Zadeh, L.A. and C.A. Desoer (1963). *Linear System Theory*, McGraw-Hill, New York.

Zeheb, E. and E. Walach (1977). 2-Parameter Root-Loci Concept and Some Applications, *IEEE International Journal of Circuit Theory and Applications*, 5, pp. 305-315.

Zeheb, E. and E. Walach (1981). Zero Sets of Multiparameter Functions and Stability of Multidimensional Systems. *IEEE Transactions on Acoustic, Speech, Signal Processing*, ASSP-29, pp. 197-206.

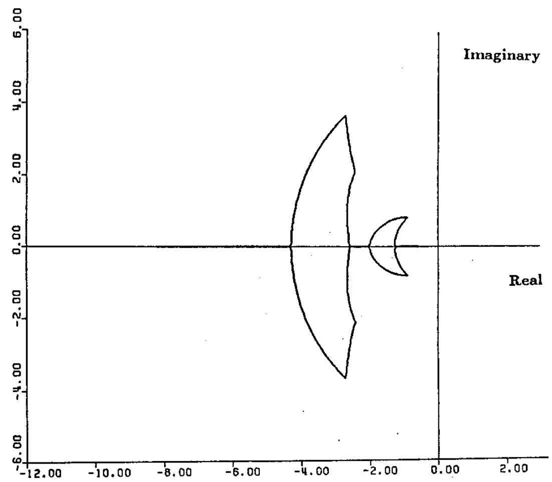

Figure 1: Cross Section of the RRL for the Low Gain Case ($k = 12$).

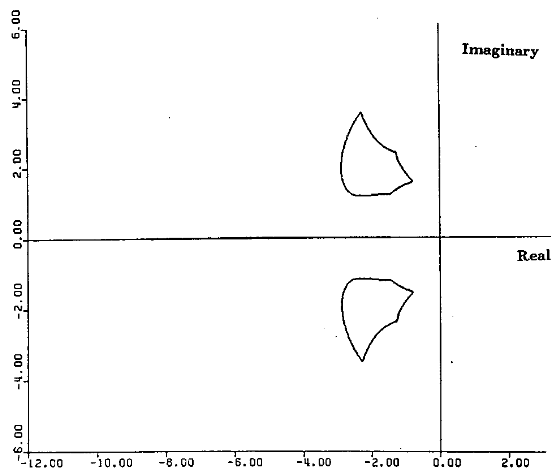

Figure 2: Cross Section of the RRL for the High Gain Case ($k = 25$).

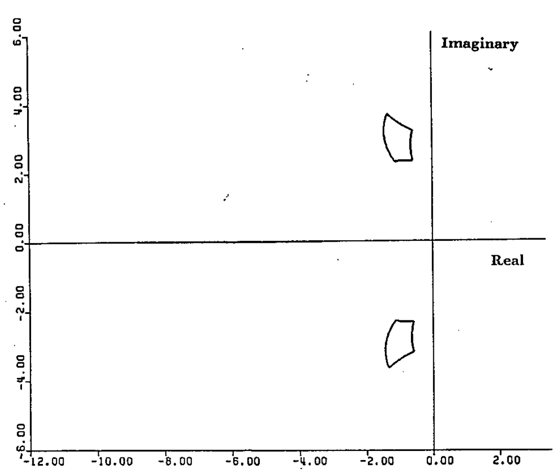

Figure 3: Cross Section of the RRL for the High Gain Case ($k = 50$).