Transient and Stability Analysis of Time-Domain Reactive Current Analyzers Using Averaged Waveforms

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Abstract—A simple method of characterizing the transient response and the stability of time-domain analog reactive current analyzers is presented. The method of analysis, based on averaged waveforms, allows one to perform a linear analysis on an otherwise nonlinear system. The ability to predict the transient characteristics of these instrumentation circuits allows the instrumentation engineer to predict the settling time of the analyzer under various waveform measurement conditions and, more importantly, detect the presence of unstable equilibrium points in these nonlinear feedback circuits.

I. INTRODUCTION

The increased use of power electronic circuits in machine drives and industrial process control is causing substantial amounts of energy to be transmitted through nonsinusoidal voltages and currents. To reduce losses in the transmission network and achieve the maximum throughput of real power to the load, it is often necessary to design compensation circuits that minimize the amount of reactive current in the network. The power engineering community is split as to whether the problem of reactive current minimization under nonsinusoidal conditions should be tackled using a frequency-domain or time-domain approach. Frequency-domain methods provide insight on how the load impedance affects the flow of nonactive (or reactive) currents in the network; however, the design of an appropriate power-factor correction circuit requires that the susceptance of the load be known at all the harmonic frequencies present in the source voltage [1]. The time-domain approach, although sometimes obscure, does not require knowledge of the load impedance, and appropriate values for shunt reactance compensators for modest power factor correction, can be determined via analog instrumentation operating on the terminal voltages and currents [2]–[6].

The analog circuits suggested in [2]–[6], are nonlinear feedback circuits due to the presence of multipliers in the feedback loop. These circuits, to date, have only been analyzed in the steady state, and their transient performances which determine the choice of loop gains for reasonable settling times and also predict the stability of these circuits are yet to be characterized. The lack of such analysis has led to circuits such as the one proposed in [3], that are inherently unstable.

In this paper, we present a simple method of analyzing the transient response and stability of these nonlinear feedback circuits that can aid designers to predict circuit performance and also assess the stability of novel circuit topologies for reactive current instrumentation. We note that the values of shunt capacitors and inductors obtained from the circuit topologies in [2]–[6] are only useful in networks with negligible source impedance, and hence in power systems with substantial inductive source impedance, these values of shunt reactors may be suboptimal [7]. However, this limitation does not detract from the method of analysis being presented here.

In this paper, we use a series LCR circuit fed by a non-sinusoidal voltage source, shown in Fig. 1, to illustrate the measurement of the equivalent load conductance \( G_e \), and hence the active or in-phase current, \( i_p \), and power factor, \( k_p \), in addition to the determination of optimum values of parallel shunt capacitors for reactive current minimization or power factor correction. We start with power factor compensation via a single shunt capacitance \( C_{opt} \) as derived in [4], [9], followed by two-element shunt parallel LC compensation as suggested in [3].

II. ASSUMPTIONS AND NOTATION

In this paper, the source voltage \( u(t) \) and source current \( i(t) \) are assumed to be periodic with no dc component, and the source impedance of the supply voltage is assumed to be negligible. The time derivative of \( u(t) \) and its integral...
are denoted by \( \dot{u} \) and \( \ddot{u} \), respectively. All rms quantities are denoted by uppercase.

Finally, we define the local average \( \bar{X}(t) \) of an instantaneous variable \( x(t) \) as
\[
\bar{X}(t) \triangleq \frac{1}{T} \int_{t-T}^{t} x(\tau) d\tau
\]
such that the derivative of the time average is equal to the time average of the derivative of the instantaneous variable. That is,
\[
\frac{d}{dt} \bar{X}(t) = \bar{\dot{X}}(t) = \frac{1}{T} \int_{t-T}^{t} \dot{x}(\tau) d\tau = \bar{\dot{x}}(t)
\]
where \( T \) is the averaging interval. If \( \bar{X}(t) \) is periodic with period \( T \) or any multiple thereof, we denote the averaged quantity as \( \langle x \rangle \). Thus, we denote average values of all periodic quantities by angle brackets and local averages of nonperiodic quantities by overbars.

III. MEASUREMENT OF ACTIVE CURRENT AND LOAD POWER FACTOR

Consider the circuit shown in Fig. 1 where a linear load is powered by a nonsinusoidal voltage source \( u(t) = \sqrt{2} \sum_n U_n \sin(n \omega_1 t) \). The admittance of the load at the \( n \)th harmonic frequency \( Y_{L_n} \) can be written as
\[
Y_{L_n} = G_{L_n} + jB_{L_n}
\]
For the circuit of Fig. 1,
\[
G_{L_n} = \frac{R(n \omega C)^2}{(n \omega RC)^2 + (n^2 \omega^2 LC - 1)^2} \quad \text{and} \quad B_{L_n} = -\frac{jn \omega C(n^2 \omega^2 LC - 1)}{(n^2 \omega^2 RC)^2 + (n^2 \omega^2 LC - 1)^2}.
\]
The source current \( i \) consists of an active (in-phase) part, \( i_p \), which is responsible for active power flow and a reactive or quadrature part \( i_q \). The two components are mutually orthogonal such that the rms value of the source current, \( I \), is given by
\[
I = \sqrt{I_p^2 + I_q^2}.
\]
In performing power factor correction, we seek to design compensation circuits that do not change \( i_p \) but minimize or eliminate \( i_q \). The active current is given by
\[
i_p(t) = \frac{P}{U^2} \cdot u(t) = G_e \cdot u(t)
\]
where \( P \) is the active power delivered to the load, and \( G_e \) is the equivalent load conductance. Notice that \( i_p \) has the same shape as the supply voltage as shown in the last plot in Fig. 3. The load power factor \( k_p \) can be written in terms of frequency-domain quantities as
\[
k_p = \frac{P}{U I} = \sum_n U_n I_n \cos(\phi_n) \frac{1}{U I} = \frac{P}{U I} = \frac{U I_p}{U I} = I_p I
\]
where \( \phi_n \) is the phase angle between the voltage and the current at the \( n \)th harmonic, or in time-domain quantities as
\[
k_p = \frac{P}{U I} = \frac{U I_p}{U I} = \frac{I_p}{I}.
\]

A. Steady-State Equilibrium Points

A block diagram of a circuit that measures the active current (and hence active power) flowing into an arbitrary load from only the terminal voltage and current is shown in Fig. 2. This circuit corresponds to the first switch position of the Kusters and Moore circuit in [2, p. 1849]. It is easily shown that in the steady-state the average value of \( X(t) \) is given by
\[
\bar{X}_{ss} = \frac{k_i}{k_u} \langle u \cdot i \rangle = \frac{k_i}{k_u} \frac{P}{U^2}
\]
where the \( ss \) subscript denotes the steady-state value [2], [3], [6]. Thus \( \bar{X}_{ss} \) represents the equivalent load conductance \( G_e \) scaled by the ratio of the gains in the voltage and current attenuators. Equation (1) can be rewritten to isolate quantities of interest, namely the active power, \( P \) and the power factor, \( k_p \), as
\[
P = \frac{k_u}{k_i} \cdot \bar{X}_{ss} \cdot U^2 \quad \text{and} \quad k_p = \frac{P}{U I} = \frac{k_u}{k_i} \cdot \bar{X}_{ss} \cdot \frac{U}{I}.
\]

IV. Transient Analysis

In order to determine the settling time of \( \bar{X}(t) \) during instrumentation and, more importantly, the stability of the equilibrium point, we need to perform a transient analysis. The loop in Fig. 2 enforces that
\[
\bar{X}(t) = k_{i_1} [k_{i_2} - k_{u_1} \cdot X(t)] k_{u_2} u(t)
\]
which leads to the nonlinear differential equation
\[
\bar{X}(t) + k_{u_1} k_{i_2}^2 [u \cdot u \cdot X(t)] = k_{u_1} k_{u_2} k_{i_1} [u \cdot i]
\]
for the state variable \( X \). To transform (4) into a constant coefficient linear differential equation, we employ the averaging procedure described in Section II and write (4) as
\[
\bar{X}(t) + \bar{k}_{u_1} \bar{k}_{i_2}^2 [u \cdot u \cdot \bar{X}(t)] = \bar{k}_{u_1} \bar{k}_{u_2} \bar{k}_{i_1} [u \cdot i].
\]
If we assume that
1) \( X(t) \) has small ripple such that over the averaging interval \( T \), \( X(t) \approx \bar{X}(t) \), and
2) \( X(t) \) varies slowly so that \( \bar{X}(t) \) does not vary significantly over the averaging interval \( T \),
Fig. 4. Shunt capacitive compensation of circuit in Fig. 1.

0.072 s. The steady-state value of $X(t)$ from Fig. 3 is 0.25. Superimposed on $X(t)$ is a plot of (8). This plot shows that the time constant and steady-state values of the system in Fig. 2 are governed by (8)

$$P = \frac{k_u}{k_i} \frac{X_{ss}}{U} = 1444.44 \text{W}$$

$$k_p = \frac{k_u}{k_i} \frac{X_{ss}}{I} = 0.316$$

and the equivalent load conductance $G_e = \frac{k_u X_{ss}}{k_i} = 0.1 \Omega$.

IV. CAPACITIVE COMPENSATION WITH A SINGLE SHUNT ELEMENT

When we seek to improve the power factor of the load via connecting a shunt capacitor across the load, as shown in Fig. 4, the optimal capacitance $C_{opt}$ has been shown using frequency-domain methods to be

$$C_{opt} = \frac{\sum n U_n^2 B_n}{\omega \sum n U_n^2 U_n}$$

(9)

Here $\omega$ is the fundamental frequency, $U_n$ is the rms value of the $n$th source voltage harmonic, and $B_n$ is the load susceptance [5], [4], [9]. For the circuit of Fig. 4, $C_{opt}$ as calculated by (9) is equal to 35.36 $\mu F$.

An equivalent expression for $C_{opt}$ in terms of time-domain quantities can be derived as follows: if the connection of a shunt capacitance $C$ across the load is to minimize the rms value of the total source current, then we require that

$$\frac{d}{dt} \left( \frac{1}{T} \int_0^T \tilde{i}_r^2 \, dt \right) = 0$$

(10)

where $\tilde{i}_r = C \tilde{u}$. If the source impedance is zero, then the connection of $C$ does not change the voltage, $\tilde{u}$, across the capacitor and the load. Thus the load current $i_L$ is unchanged, and hence the derivative of $\tilde{u}$ and $i_L$ with respect to $C$ are zero. Therefore, (10) can be written as

$$\frac{1}{T} \int_0^T \tilde{i}_L(C \tilde{u} + i_L) \, dt = 0, \Rightarrow \frac{1}{T} \int_0^T \tilde{u} \cdot i_L \, dt = -\frac{1}{T} \int_0^T C \tilde{u}^2 \, dt.$$
Thus

\[ C = C_{\text{opt}} = -\frac{k_i}{k_u} \langle u \cdot \dot{u} \rangle \quad \text{and} \quad \dot{Y}_{ss} = \frac{k_i}{k_u} \langle \dot{u} \cdot \dot{u} \rangle \]  

(11)

where we have replaced \( i_L \) with \( i \); the original source current before the connection of the capacitor. It can be shown that (11) and (9) yield identical results [4].

Fig. 5 shows a block diagram of a circuit that determines the value of \( C_{\text{opt}} \) using time-domain waveforms. This block diagram corresponds to the second switch position presented in [2].

Fig. 6 shows the transient response of the circuit in Fig. 5 for \( k_u = 0.08 \), \( k_i = 0.2 \), \( k_{\text{inu}} = 9.3 \), \( k_{\text{inu}} = 0.15 \), \( k_{\text{inu}} = 0.2 \), \( U = (100/3) \cdot \sqrt{3} \) V and \( U = 8.43 \times 10^{-3} \) V/s; yielding \( \tau_u = 0.072 \) s and \( \tau_u = 0.087 \) s. The steady-state value of \( \dot{X}(t) \) remains unchanged as in the previous simulation and that of \( \dot{Y}(t) \) is \(-0.0786\). Superimposed on the \( \dot{X}(t) \) and \( \dot{Y}(t) \) waveforms are their average values as predicted by

\[ \dot{X}(t) = k_i \frac{\langle u \cdot \dot{u} \rangle}{U^2} [1 - e^{-t/\tau_u}] \]

and

\[ \dot{Y}(t) = k_i \frac{\langle \dot{u} \cdot \dot{u} \rangle}{U^2} [1 - e^{-t/\tau_u}] \]  

(14)

where \( \tau_u = 1/k_u k_i^2 U^2 \) and \( \tau_u = 1/k_u k_i^2 U^2 \). Inductive shunt compensation can also be achieved in a similar manner by replacing \( \dot{u} \) with the integral \( \dot{u} \) of the source voltage.

Once again, the time constants are inversely proportional to the size of the inputs. Therefore, in order to ensure that the circuit settles in a reasonable time when waveforms with small rms values are being analyzed, automatic gain controls (AGC's) may be needed to automatically adjust the amplitudes of \( \dot{u} \) and \( \dot{u} \). The AGC's are also needed if the circuit is to be used to analyze waveforms over a wide frequency range, since the outputs of the operational amplifiers used to compute \( u \) and \( u \) are proportional to the frequency of the input waveform.

Fig. 6 shows the transient response of the circuit in Fig. 5 for \( k_u = 0.08 \), \( k_i = 0.2 \), \( k_u = 9.3 \), \( k_{\text{inu}} = 0.15 \), \( k_{\text{inu}} = 0.2 \), \( U = (100/3) \cdot \sqrt{3} \) V and \( U = 8.43 \times 10^{-3} \) V/s; yielding \( \tau_u = 0.072 \) s and \( \tau_u = 0.087 \) s. The steady-state value of \( \dot{X}(t) \) remains unchanged as in the previous simulation and that of \( \dot{Y}(t) \) is \(-0.0786\). Superimposed on the \( \dot{X}(t) \) and \( \dot{Y}(t) \) waveforms are their average values as predicted by

\[ C_{\text{opt}} = -\frac{k_u}{k_i} \cdot Y_{ss} = 35.36 \mu F \]  

(15)

which is in agreement with (9). We note that the power factor with shunt capacitive compensation only increased slightly; from 0.32 to 0.45.

V. PARALLEL TWO-ELEMENT SHUNT LC COMPENSATION

It is possible to further improve the power factor of the circuit in Fig. 1 by using a parallel combination of a shunt capacitor and a shunt inductor, as shown in Fig. 7. Since the \( LC \) resonant frequency is between the two harmonics in the source voltage, it is possible to make the circuit resonant at
both harmonic frequencies. In the frequency domain the values of the appropriate shunt inductance \( L_s \) and shunt capacitance \( C_s \) can be found by solving two simultaneous equations that make the imaginary part of the combined shunt admittance of the load and compensator vanish at both harmonic frequencies. This analysis is straightforward and will not be repeated here. The results from such analysis yield \( L_s = 8.841 \text{ mH} \) and \( C_s = 265.25 \text{ pF} \).

It is shown in [3] that to minimize the rms value of the source current, \( L_s \) and \( C_s \) must satisfy the following two time-domain equations simultaneously:

\[
\frac{(u \cdot u)}{L_s} - C_s \cdot (\ddot{u} \cdot \dot{u}) = (\ddot{u} \cdot i) \tag{16}
\]

\[
C_s \cdot (u \cdot u) - \frac{(\ddot{u} \cdot \dot{u})}{L_s} = (\ddot{u} \cdot i). \tag{17}
\]

Solving the preceding two equations yield,

\[
L_s = \frac{(u \cdot u)^2 - (\ddot{u} \cdot \dot{u})(\ddot{u} \cdot \dot{u})}{(u \cdot u)(\ddot{u} \cdot \dot{u}) + (\ddot{u} \cdot \dot{u})(\ddot{u} \cdot \dot{u})} \tag{18}
\]

and

\[
C_s = \frac{(u \cdot u)(\ddot{u} \cdot \dot{u}) + (\ddot{u} \cdot \dot{u})(\ddot{u} \cdot \dot{u})}{(u \cdot u)^2 - (\ddot{u} \cdot \dot{u})(\ddot{u} \cdot \dot{u})} \tag{19}
\]

Note that (18) and (19) are written in terms of time-domain quantities obtainable from the source terminal voltage and current waveforms.

In [3], it was shown that the circuits with the block diagrams shown in Figs. 8 and 10 can be used to automatically compute the values of \( C_s \) and \( L_s \). Although both circuits have identical steady-state equilibrium points, a transient analysis shows that the circuit in Fig. 10 is unstable.

\[\text{Fig. 8. Block diagram of circuit that solves (16) and (17).}\]

The loops in Fig. 8 ensure that

\[
\begin{align*}
\hat{A}(t) &= k_a[k_i \ddot{u} - k_s \dot{u} \cdot A(t)] - k_s \dot{u} \cdot B(t)k_a \ddot{u} \\
&= k_a[k_i \ddot{u} + k_s \dot{u} \cdot A(t)] - k_s \dot{u} \cdot B(t)k_a \ddot{u} \\
\hat{B}(t) &= k_b[k_i \ddot{u} - k_s \dot{u} \cdot A(t)] - k_s \dot{u} \cdot B(t)k_a \ddot{u} \\
&= k_b[k_i \ddot{u} + k_s \dot{u} \cdot A(t)] - k_s \dot{u} \cdot B(t).
\end{align*}
\]

which, after averaging, can be written in matrix form as

\[
\begin{bmatrix}
\hat{A}(t) \\
\hat{B}(t)
\end{bmatrix} =
\begin{bmatrix}
\frac{k_a k_s^2}{3} (\ddot{u} \cdot \dot{u}) & \frac{k_a k_s}{3} (\ddot{u} \cdot \dot{u}) & \frac{k_a}{3} (\ddot{u} \cdot \dot{u}) & \frac{k_a}{3} (\ddot{u} \cdot \dot{u}) \\
k_b & k_b & k_b & k_b
\end{bmatrix} -
\begin{bmatrix}
\frac{k_s k_i^2}{3} (\ddot{u} \cdot \dot{u}) & \frac{k_s k_i}{3} (\ddot{u} \cdot \dot{u}) & \frac{k_s}{3} (\ddot{u} \cdot \dot{u}) & \frac{k_s}{3} (\ddot{u} \cdot \dot{u}) \\
k_b & k_b & k_b & k_b
\end{bmatrix} \\
\begin{bmatrix}
\frac{k_s k_i^2}{3} (\ddot{u} \cdot \dot{u}) & \frac{k_s k_i}{3} (\ddot{u} \cdot \dot{u}) & \frac{k_s}{3} (\ddot{u} \cdot \dot{u}) & \frac{k_s}{3} (\ddot{u} \cdot \dot{u}) \\
k_b & k_b & k_b & k_b
\end{bmatrix} \begin{bmatrix}
\hat{A}(t) \\
\hat{B}(t)
\end{bmatrix} \tag{20}
\]

Noting that \((\ddot{u} \cdot \dot{u}) = -U^2\) for periodic waveforms with no dc component, we can write (20) as

\[
\begin{bmatrix}
\hat{A}(t) \\
\hat{B}(t)
\end{bmatrix} =
\begin{bmatrix}
\frac{k_a k_s^2}{3} (\ddot{u} \cdot \dot{u}) & \frac{k_a k_s}{3} (\ddot{u} \cdot \dot{u}) & \frac{k_a}{3} (\ddot{u} \cdot \dot{u}) & \frac{k_a}{3} (\ddot{u} \cdot \dot{u}) \\
k_b & k_b & k_b & k_b
\end{bmatrix} -
\begin{bmatrix}
\frac{k_s k_i^2}{3} (\ddot{u} \cdot \dot{u}) & \frac{k_s k_i}{3} (\ddot{u} \cdot \dot{u}) & \frac{k_s}{3} (\ddot{u} \cdot \dot{u}) & \frac{k_s}{3} (\ddot{u} \cdot \dot{u}) \\
k_b & k_b & k_b & k_b
\end{bmatrix} \\
\begin{bmatrix}
\frac{k_s k_i^2}{3} (\ddot{u} \cdot \dot{u}) & \frac{k_s k_i}{3} (\ddot{u} \cdot \dot{u}) & \frac{k_s}{3} (\ddot{u} \cdot \dot{u}) & \frac{k_s}{3} (\ddot{u} \cdot \dot{u}) \\
k_b & k_b & k_b & k_b
\end{bmatrix} \begin{bmatrix}
\hat{A}(t) \\
\hat{B}(t)
\end{bmatrix} \tag{21}
\]
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Source Voltage \[u(t)\]

\[
\begin{array}{c}
\text{Time (s)} \\
0 \ 0.05 \ 0.1 \ 0.15 \ 0.2 \ 0.25 \ 0.3 \ 0.35 \ 0.4 \ 0.45 \ 0.5
\end{array}
\]

Source Current \[i(t)\]

\[
\begin{array}{c}
\text{Time (s)} \\
0 \ 0.05 \ 0.1 \ 0.15 \ 0.2 \ 0.25 \ 0.3 \ 0.35 \ 0.4 \ 0.45 \ 0.5
\end{array}
\]

\[\text{Fig. 9. Transient response of the system in Fig. 8.}\]

\[\text{Fig. 10. Alternative circuit that solves (16) and (17).}\]

\[
\begin{bmatrix}
-a & b \\
c & -d
\end{bmatrix}
\begin{bmatrix}
\dot{A}(t) \\
\dot{B}(t)
\end{bmatrix}
+ \begin{bmatrix}
k_a k_i k_j k_l (\ddot{u} \cdot \dddot{i}) \\
k_b k_i k_j k_l (\dddot{u} \cdot \dddot{i})
\end{bmatrix}
\begin{bmatrix}
\dot{A}(t) \\
\dot{B}(t)
\end{bmatrix}
\]

where \(a, b, c,\) and \(d\) are positive numbers. This system, unlike those in Section IV, is not decoupled. In the steady state, where \(\dot{A}(t)\) and \(\dot{B}(t)\) are zero, (21) and (22) yield

\[
\begin{align*}
\dot{A}_{ss} &= -\frac{k_i}{k_a} \frac{\langle \dddot{u} \cdot \dddot{i} \rangle + \langle \dddot{u} \cdot \dddot{i} \rangle}{\langle \dddot{u} \cdot \dddot{u} \rangle} \\
\dot{B}_{ss} &= -\frac{k_i}{k_a} \frac{\langle \dddot{u} \cdot \dddot{u} \rangle + \langle \dddot{u} \cdot \dddot{u} \rangle}{\langle \dddot{u} \cdot \dddot{u} \rangle}
\end{align*}
\]

Thus, from (18) and (19)

\[
C_u = -\frac{k_i}{k_a} \cdot \dot{A}_{ss} \quad \text{and} \quad L_u = -\frac{k_i}{k_a} \cdot \dot{B}_{ss}. \quad (25)
\]

Using the Hurwitz criterion, we deduce that the system described by (22) is stable if \(ad > bc\). That is

\[
k_a k_b k_j k_l U^2 > k_a k_b k_j k_l U^2 - k_a k_b k_j k_l U^2 \Rightarrow U > U \cdot U. \quad (26)
\]

For a nonsinusoidal voltage \(u\) given by

\[
u = \sum_{n=1}^{N} \sqrt{2} U_n \sin(n \omega t + \alpha_n)
\]

it can be shown that

\[
\frac{U}{\ddot{U}} < \frac{U}{\ddot{U}} \Rightarrow \dot{U} \cdot U > U \cdot U. \quad (28)
\]

Thus, the inequality in (27) is always satisfied, and the system is always stable. Furthermore, for the condition given in (27), the eigenvalues of the system matrix are never complex. Therefore, the system response is never oscillatory for waveforms with constant \(\langle uu \rangle\) and \(\langle uu \rangle\).

Fig. 9 shows the simulated response of the circuit in Fig. 8 to a step input in power \(P\) for a nonsinusoidal input voltage. For this simulation, for \(k_a = 0.00008, k_i = 0.2, k_b = 8, k_d = 4, k_b = 5, U = (100/3) \cdot \sqrt{13} \quad \text{V}, \quad U = 8.43EE4 \quad \text{V/s, and} \quad \ddot{U} = 0.2717 \quad \text{V/s. The eigenvalues} \lambda_1, \lambda_2 \quad \text{of the system}
matrix are \( \lambda_1 = -192.07 \) and \( \lambda_2 = -13.48 \). The evolution of the average values of the state \( A \) and \( B \) is given by

\[
\dot{A}(t) = A_{ss} + [C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}]
\]

and

\[
\dot{B}(t) = B_{ss} + [C_3 e^{\lambda_1 t} + C_4 e^{\lambda_2 t}],
\]

where \( A_{ss} \) and \( B_{ss} \) are given by (23) and (24), respectively, and

\[
\begin{align*}
C_1 &= k_a k_3 k_5 (\bar{u} \cdot \bar{i}) + \lambda_2 A_{ss} \\
C_2 &= k_a k_3 k_5 (\bar{u} \cdot \bar{i}) + \lambda_1 A_{ss} \\
C_3 &= k_b k_3 k_5 (\bar{u} \cdot \bar{i}) + \lambda_2 B_{ss} \\
C_4 &= k_b k_3 k_5 (\bar{u} \cdot \bar{i}) + \lambda_1 B_{ss}
\end{align*}
\]

The plots of \( \dot{A}(t) \) and \( \dot{B}(t) \) are also superimposed on the simulation waveforms in Fig. 9. We see that the system is stable, and that the average values of the state variables evolve as predicted. Using the final values of \( \dot{A}(t) \) and \( \dot{B}(t) \) we calculate \( C_s = 265.25 \mu F \) and \( L_s \) as predicted. Using the final values of \( \dot{A}(t) \) and \( \dot{B}(t) \) we calculate \( C_s = 265.25 \mu F \) and \( L_s = 8.841 \) mH from (25) by multiplication of \( A_{ss} \) and \( B_{ss} \) by the appropriate scaling factors.

With the values of shunt capacitance and inductance as calculated, the power factor of the circuit in Fig. 7 is now unity. In general, unity power factor may not be achieved unless the load conductance at all the harmonic frequencies is equal to the equivalent load conductance \( G_e \) [1], as is the case here.

In [3], it was also shown that the block diagram in Fig. 10 solved (16) and (17), and hence could be used to compute \( C_s \) and \( L_s \). This circuit corresponds to Fig. 8 except that the inputs to the second multiplier, \( \bar{u} \) and \( \bar{i} \), are switched.

The differential equations for the averaged state variables are given by

\[
\begin{bmatrix}
\dot{A}(t) \\
\dot{B}(t)
\end{bmatrix} = \begin{bmatrix}
k_a k_3 k_5 \bar{u} \cdot \bar{i} & k_a k_5 (\bar{u} \cdot \bar{i}) \\
k_b k_3 k_5 \bar{u} \cdot \bar{i} & k_b k_5 (\bar{u} \cdot \bar{i})
\end{bmatrix} \begin{bmatrix}
A(t) \\
B(t)
\end{bmatrix}
\]

(31)

Again, noting that \( (\bar{u} \bar{i}) = -U^2 \) for periodic waveforms with no dc component, we can write (20) as

\[
\begin{bmatrix}
\dot{A}(t) \\
\dot{B}(t)
\end{bmatrix} = \begin{bmatrix}
k_a k_3 k_5 U^2 & -k_a k_5 (\bar{u} \cdot \bar{i}) \\
-k_b k_3 k_5 (\bar{u} \cdot \bar{i}) & k_b k_5 U^2
\end{bmatrix} \begin{bmatrix}
A(t) \\
B(t)
\end{bmatrix}
\]

(32)

For a stable system we require that

\[
k_a k_3 k_5 U^2 < k_a k_5 (\bar{u} \cdot \bar{i}) < k_b k_3 k_5 U^2
\]

\[
(33)
\]

\[
\Rightarrow \bar{U} \cdot \bar{U} < U \cdot U
\]

(34)

Equation (34) is the complement of (28); therefore this system is unstable as demonstrated by the step response in Fig. 11.

VI. CONCLUSION

Waveform averaging techniques can be used to transform the nonlinear models for the time-domain analog reactive current analyzers presented in [2]–[4] into linear differential equations with constant coefficients. Time-domain current analyzers provide useful information about the load, such as the active power supplied, reactive power and power factor, using only measurements of the terminal voltages and currents.

The waveform averaging technique presented here results in a linear model that is easily analyzed, allowing the circuit designer to predict the transient response and stability of the nonlinear feedback circuit. We have used this methodology in conjunction with an LCR load fed by a nonsinusoidal voltage, to analyze and confirm the transient behavior of time-domain current analyzers presented in [2]–[4], and in the process, have shown instability in one of the circuits presented in [3]. The technique employed here can also be used to characterize the transient behavior of the reactive current converter presented in [6] and other such nonlinear feedback circuits.

REFERENCES

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